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An implicit cell-centered Lagrange-Remap scheme for all speed flows

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ABSTRACT

A second-order time-accurate implicit scheme is constructed for the Lagrange-Remap (LR) strategy. The numerical flux is given by the simple acoustic Riemann solver. The Riemann solver is modified by introducing a scaling coefficient so that the scheme can deal with very subsonic (low Mach) flows as well as supersonic flows. The Lagrange step solves implicitly the hyperbolic equations of pressure and velocities under the isentropic assumption by the trapezoidal time integration method. The new LR scheme maintains exactly the conservation of mass, momentum and energy, and it is general for materials with any equation of state. Numerical tests show that the LR scheme using the simple but general Riemann solver can resolve shock waves sharply for supersonic flows, and resolve well the acoustic waves in low Mach number flows as low as $M = 0.001$.

1. Introduction

The upwind schemes, proved to be able to monotonely capture a discontinuity with the minimum amount of artificial viscosity for the 1-D scalar hyperbolic equation, have gained great acceptance in industrial applications and academic studies [1], especially for compressible flows associated shock waves. The Euler equations consist of two acoustic waves that propagate typically at the sound speed, and other waves (contact and shear waves) that move at the speed of fluid particle. The realization of a upwind discretization for the system of the Euler equations is not simple, since the waves are generally not unidirectional. The Godunov-type approach solves this problem by pursuing an exact or approximate solution to wave interactions, while the flux vector splitting approach decomposes the system such that each subsystem is unidirectional. For multi-dimensional Euler equations, the extension based on the 1D Riemann upwind solvers, which neglects the contribution of shear waves, contains a large amount of empiricism and must therefore remain suspect, although these schemes have been successfully applied to practical problems. Many Godunov-type schemes contain subtle flaws that can cause spurious solutions [2].

On the other hand, the acoustic waves are an essential ingredient in compressible flows, but they even do not explicitly appear in incompressible flows. Undesirable effects of low Mach number flows on an upwind scheme without any modification include low convergence speed and loss of accuracy [3]. There are at least two techniques to solve the problem. One is the preconditioning method that performs a matrix operation on the original Jacobian matrix such that the coefficients of numerical viscosity terms are of the similar magnitude in the low Mach limit (e.g. [4,5]). Another method directly rescales the coefficients of numerical viscosity in the flux vector splitting [7–9] and the Riemann solvers [10–12] among many others. It is clear that the acoustic waves should be treated differently at least in the low Mach limit, preferably treated separately from the other waves that move at the flow velocity.

The LR two-step method is able to separate the acoustic waves from other waves. The method is closely related to the arbitrary Lagrangian–Eulerian (ALE) method [13]. The ALE provides a general framework that can be used to combine the best properties of Lagrangian and Eulerian methods. Brief reviews of recent Lagrange schemes and remapping methods can be found in [14,15]. If the solution is remapped onto the original Eulerian (spatially fixed) mesh without rezoning, it behaves as the Eulerian methods on a fixed mesh. Although the idea of this simple LR strategy was proposed in [16] long before the appearance of ALE, it is often regarded as a special case of ALE (e.g. fixed-mesh ALE [17]). The LR strategy can be realized without explicitly dealing with node movements, while maintaining the advantages of the Lagrangian schemes. In the pioneer work of van Leer[18], a second-order LR scheme was constructed via piecewise linear reconstruction, known as the MUSCL approach, in which the exact Riemann solver was used, and the 2D extension was realized by dimensional splitting.

The approximate Riemann solver can be used in the Lagrange step. Rider [19] investigated and compared the behavior of a few approximate Riemann solvers in the first-order LR schemes. All first-order schemes suffer from over diffusivity more or less. With

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the help of monotonicity-preserving anti-diffusion mechanism, the contact and the material interfaces can be resolved sharply in one grid cell [20,21], although the overall accuracy remains first-order accurate. There is a renewed interest in applying the strategy for compressible multiphase flows[21,22]. An anti-diffusive scheme was developed for a five-equation model [21]. The second-order LR strategy is used instead for the compressible multiphase flows following the MUSCL approach, and two-dimensional extension is again based on dimensional splitting and restricted to the Cartesian mesh [22]. In principle, higher orders may be achieved by polynomial reconstructions [23].

In 7th ICCFD, we formulated the LR scheme as the finite volume method so that it can be implemented for unstructured grids without recourse to dimensional splitting, and proposed a second-order explicit scheme based on the acoustic Riemann solver [25], the simplest one that may resolve a contact discontinuity. In this paper, the implicit treatment of the acoustic waves is reported. It turns out that the compressible and almost incompressible flows ranging from very subsonic to supersonic can be resolved using the acoustic Riemann solver with a simple modification in the LR scheme.

2. Finite volume LR formulation for the Euler equations

Consider the finite volume discretization of the conservation laws,

$$\Omega_i \mathbf{U}_i^{n+1} = \Omega_i \mathbf{U}_i^n - \Delta t \sum_k \mathbf{F}_{ik}^* S_{ik}, \tag{1}$$

where $\mathbf{F}_k$ is the vector of numerical flux at volume face k with the length in 2D or the area in 3D, S_{ik}. Symbols Δt and Ω_i are the time step and the volume in the fixed Eulerian grid respectively. The flux vector is basically

$$\mathbf{F} = u_n \mathbf{U} + \mathbf{P}, \tag{2}$$

where conservative quantities $\mathbf{U}$ are $(\rho, \rho\mathbf{u}, \rho E)^T$, and $\mathbf{P} = (0, p\mathbf{n_s}, pu_n)^T$, where $\mathbf{n}_s$ is the unit outward normal vector, and normal velocity $u_n = \mathbf{u}\cdot\mathbf{n}_s$. The specific total energy contains the specific internal energy and kinetic energy, $E = e + \mathbf{u}\cdot\mathbf{u}/2$. The conservation laws must be supplemented by the equation of state (EOS), $p = p(\rho, e)$. The numerical method to be introduced is designed for any fluid, either gas or liquid; no parameter optimization or tuning will be made for specific EOS, such as perfect gases.

In the LR framework, the conservative scheme (1) is solved in a two-step fashion as follows,

$$\widetilde{\Omega}_i \widetilde{\mathbf{U}}_i = \Omega_i \mathbf{U}_i^n - \Delta t \sum_k \mathbf{P}_{ik}^* S_{ik}, \tag{3}$$

$$\Omega_i \mathbf{U}_i^{n+1} = \widetilde{\Omega}_i \widetilde{\mathbf{U}}_i - \Delta t \sum_k \left(u_n^* \widetilde{\mathbf{U}}^*\right)_{ik} S_{ik}, \tag{4}$$

which correspond to the Lagrange step and the remap step respectively. The symbols with tilde denote the intermediate quantities in the Lagrange step, in order to distinguish them from those in the next time step. One may recover the original finite volume formulation (1) immediately by adding two relations (3) and (4). The symbols with asterisk are defined in the numerical flux. Pressure p^* and velocity u_n^* are to be given by the acoustic Riemann solver [24]. The volume of the cell in the Lagrangian frame follows:

$$\widetilde{\Omega}_i = \Omega_i + \Delta t \sum_k (u_n^*)_{ik} S_{ik}. \tag{5}$$

The state of the conservative quantities $\widetilde{\mathbf{U}}^*$ are interpolated from those updated by (3) and (5). The remap step (4) is simply the geometric remap procedure, which is the same for both explicit and implicit schemes. A geometric interpretation of the remap step is given in [25].

2.1. Implicit Lagrange step

The explicit scheme has been reported and evaluated in [25], and this work will focus on the implicit implementation of (3). Since the change of the conservative quantities of a Lagrangian cell is solely determined by pressure and velocity at grid face, as seen from the elements of flux vector $\mathbf{P}$ in the numerical flux of (3), it is reasonable to choose primitive variables. Consider the conservation of momentum in the Lagrangian frame,

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p = 0, \tag{6}$$

and the conservation of energy under the isentropic assumption,[1]

$$\frac{\rho}{I^2} \frac{Dp}{Dt} + \nabla \cdot \mathbf{u} = 0, \tag{7}$$

where $I = \rho c$ is the acoustic impedance with isentropic sound speed $c^2 = \left(\frac{Dp}{D\rho}\right)_s$. Note that the continuity equation becomes the same as (7) in the Lagrangian frame. Performing the time integration for the finite volume formulation of a control volume with mass, $m_i = \rho_i \Omega_i$, one obtains

$$m_i \frac{\tilde{\mathbf{u}}^{n+1} - u^n}{\Delta t} + \beta \sum_k p_{ik}^{*(n+1)} \mathbf{S}_{ik} + (1-\beta) \sum_k p_{ik}^{*(n)} \mathbf{S}_{ik} = 0, \tag{8}$$

$$\frac{m_i}{I_i^2} \frac{\tilde{p}^{n+1} - p^n}{\Delta t} + \beta \sum_k \mathbf{u}_{ik}^{*(n+1)} \cdot \mathbf{S}_{ik} + (1-\beta) \sum_k \mathbf{u}_{ik}^{*(n)} \cdot \mathbf{S}_{ik} = 0. \tag{9}$$

It is the standard backward Euler method for $\beta = 1$, and the trapezoidal method ($\beta = 0.5$) is used throughout the paper for achieving second-order accuracy in time. Pressure $\tilde{p}^{n+1}$ and velocity $\tilde{\mathbf{u}}^{n+1}$ are the intermediate quantities in the Lagrange step, so they are marked with the tilde in order to distinguish them from those in the next time step p^{n+1} and $\mathbf{u}^{n+1}$. The outward surface normal, $\mathbf{S}_{ik} = S_{ik}\mathbf{n_s}$, is defined on the Eulerian grid. Following the Godunov-type approach, the quantities at grid interface (pressure p^* and velocity u_n^*) are given by the Riemann solution. The well-known acoustic Riemann solver is adapted for all speed flows,

$$p^* = \frac{I^R p^L + I^L p^R}{I^L + I^R} + f_a \frac{I^L I^R}{I^L + I^R} (u_n^L - u_n^R), \tag{10}$$

$$u^* = \frac{I^L u_n^L + I^R u_n^R}{I^L + I^R} + \frac{1}{f_a} \frac{1}{I^L + I^R} (p^L - p^R), \tag{11}$$

where $I^L = \rho^L c^L$, and $I^R = \rho^R c^R$ are acoustic impedances on two sides. (p^L, u_n^L) and (p^R, u_n^R) are pressure and normal velocity on two sides of the grid interface. Note that the acoustic Riemann solver is the exact solution to the wave pattern of two acoustic waves with a contact or a material interface between; it might be the simplest approximate Riemann solver for fluids with any EOS.

The Riemann solver is identical to the classic acoustic Riemann solver when $f_a = 1$. The acoustic Riemann solver, which assumes the wave configuration of two sound waves with an interface between, is a special case of a more general solution derived in [24] for the Lagrange scheme. The first term on the right-hand side of (10) and (11) corresponds to the weighted average of pressure and velocity respectively; it becomes the simple algebraic average in the low Mach limit. The second term corresponds to the artificial viscosity term that is required to maintain the stability of an upwind scheme, and to suppress possible numerical oscillations around a sharp discontinuity in compressible flows. Notice that the coefficient of numerical viscosity term in (10) is of the order of $O(c)$ and that in (11) is of $O(1/c)$. In the incompressible limit,

[1] The isentropic assumption does not imply that the proposed Lagrange scheme is valid only for isentropic flows. The use of isentropic sound speed is common in the analysis and design of numerical techniques for compressible flows with shock waves, for instance.