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Moving least squares simulation of free surface flows

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ABSTRACT

In this paper a Moving Least Squares method (MLS) for the simulation of 2D free surface flows is presented. The emphasis is on the governing equations, the boundary conditions, and the numerical implementation. The compressible viscous isothermal Navier–Stokes equations are taken as the starting point. Then a boundary condition for pressure (or density) is developed. This condition is applicable at interfaces between different media such as fluid–solid or fluid–void. The effect of surface tension is included. The equations are discretized by a moving least squares method for the spatial derivatives and a Runge–Kutta method for the time derivatives. The computational frame is Lagrangian, which means that the computational nodes are convected with the flow. The method proposed here is benchmarked using the standard lid driven cavity problem, a rotating free surface problem, and the simulation of drop oscillations. A new exact solution to the unsteady incompressible Navier–Stokes equations is introduced for the rotating free surface problem.

1. Introduction

Modelling flows with a free surface is a non-trivial task because the fluid domain changes during the simulation. In the VOF method [1] the liquid and the gas phase is modelled on a fixed grid of nodes (Eulerian approach). This means that the fluid–gas interface is *captured* in an average sense by some of the grid cells. In the Level-Set Method [2] the boundary is given implicitly as a level curve of a scalar field defined over the entire solution domain. This is also a Eulerian approach with approximate representation of the free surface. Another class of methods exists in which Lagrangian coordinates are used. One example is the SPH method [3] in which particles interact with each other in a pair-wise fashion and convect with the flow.

The method proposed here is based on the Reproducing Kernel Method [4] which is a kind of gridless finite difference method. It is also a Lagrangian approach and allows for explicit *tracking* of the free surface using special surface nodes. A downside is the non-uniform distribution of the nodes as the simulation progresses. Therefore, redistribution of the computational nodes is necessary at certain intervals. The basis functions used for the spatial derivatives can also be used for the interpolation of the current field values onto the new node set. MLS is basically a *centered* scheme, which has a drawback regarding decoupling modes. However, the redistribution of nodes has a smoothening effect, which in many cases helps to keep the unwanted modes under control.

The main benefits of the proposed method are:

(i) In a two-phase flow problem where one phase can be considered void (with a hydrostatic pressure) the computational domain needs only include the other phase. In such cases the simulation problem becomes smaller than e.g. VOF and Level-Set Method simulations.
(ii) No global system of equations needs to be solved.
(iii) The method can handle large deformations without the need to redefine a computational mesh.

2. Governing equations

The Navier–Stokes equations express the rate of change of momentum for a fluid particle. Using tensor notation and Cartesian coordinates they may be stated as [5]

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \tag{1}$$

where $\mathrm{d}(\cdot)/\mathrm{d}t$ denotes the material derivative with respect to time, and

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{2}$$

is the stress tensor. The symbol δ_{ij} is the Kronecker delta, and the shear stress τ_{ij} is given by

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$$\tau_{ij} = \mu \varepsilon_{ij} \tag{3}$$

where μ is the dynamic viscosity, and ε_{ij} is the deformation rate tensor

$$\varepsilon_{ij} = \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij} \tag{4}$$

For a 2D problem equations Eqs. (1)–(4) give the Navier–Stokes equations for a compressible fluid with constant viscosity

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{\rho}\left[-\frac{\partial p}{\partial x} + \mu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right)\right] \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{\rho}\left[-\frac{\partial p}{\partial y} + \mu\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right)\right] \end{aligned} \tag{5}$$

In Eqs. (1) and (5) the left hand side of the momentum equations is written using the material derivative. This notation indicates that Lagrangian coordinates are used. Lagrangian coordinates are associated with points that follow the deformation of the media which is expressed by the following equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v \tag{6}$$

The continuity equation expresses the conservation of mass, and may be written in Lagrangian form as [5]

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{7}$$

The present model assumes isothermal flow and the system is closed using the ideal gas law

$$p = C_0 \rho \tag{8}$$

where C_0 is a constant. The speed of sound a relates to the pressure–density function in the following way

$$a^2 = \frac{\partial p}{\partial \rho} = C_0 \tag{9}$$

Thus the speed of sound is controlled directly by the gas constant.

3. Boundary conditions

For the problems considered two types of boundary conditions are needed, namely a solid wall and a free surface.

3.1. Solid wall

On solid bodies the no-slip condition is used. The physical meaning is that there can be no relative velocity between particles on the solid and fluid particles touching the solid. Therefore, the velocity field of the fluid must equal the velocity on the solid at the solid/fluid interface

$$u_{\text{fluid}} = u_{\text{wall}} \quad v_{\text{fluid}} = v_{\text{wall}} \tag{10}$$

In order to evaluate stresses and forces on solid walls the pressure on the solid boundary is needed (see Eq. (2)).

A boundary condition for pressure (or equivalently density) is derived from the momentum equations Eq. (5) and the normal vector on the solid $\mathbf{n} = (n_x\ n_y)^T$. Taking the scalar product between these vectors gives

$$\begin{aligned} n_x\frac{\mathrm{d}u}{\mathrm{d}t} + n_y\frac{\mathrm{d}v}{\mathrm{d}t} &= n_x\frac{1}{\rho}\left[-\frac{\partial p}{\partial x} + \mu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right)\right] \\ &\quad + n_y\frac{1}{\rho}\left[-\frac{\partial p}{\partial y} + \mu\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right)\right] \end{aligned} \tag{11}$$

Using the notation $\partial p/\partial \mathbf{n} = n_x \partial p/\partial x + n_y \partial p/\partial y$ Eq. (11) may be rearranged in the following way

$$\begin{aligned} \frac{\partial p}{\partial \mathbf{n}} &= -\rho\left(n_x\frac{\mathrm{d}u}{\mathrm{d}t} + n_y\frac{\mathrm{d}v}{\mathrm{d}t}\right) + n_x\left[\mu\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right)\right] \\ &\quad + n_y\left[\mu\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right)\right] \end{aligned} \tag{12}$$

which constitutes a Neumann boundary condition for pressure p.

The partial derivatives of pressure in the left hand side of Eq. (12) can be replaced by expressions obtained from the equation of state Eq. (12)

$$\frac{\partial p}{\partial x} = C_0\frac{\partial \rho}{\partial x} \quad \frac{\partial p}{\partial y} = C_0\frac{\partial \rho}{\partial y} \tag{13}$$

With this substitution Eq. (12) becomes a single inhomogeneous Neumann condition for the single unknown ρ.

3.2. Free surface

The condition which must be satisfied at a free surface is the balance of normal and shear stresses. A free surface is sketched in Fig. 1. The traction at the fluid side of the free surface interface is composed of two terms. One term stems from the stress tensor associated with the fluid, the other term is due to surface tension. Surface tension gives rise to a jump in normal stress f, which is proportional to the curvature κ of the free surface

$$f = \kappa\sigma = \frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}}\sigma \tag{14}$$

where σ is the *surface tension* and x and y describes the free surface expressed in terms of the arc length parameter τ.

A boundary condition for the free surface is obtained by requiring a balance of the normal and shear stress at either side of the interface

$$\begin{aligned} \sigma^{\mathrm{A}}_{ij} n_i n_j + \kappa\sigma &= \sigma^{\mathrm{B}}_{ij} n_i n_j \\ \sigma^{\mathrm{A}}_{ij} n_i t_j &= \sigma^{\mathrm{B}}_{ij} n_i t_j \end{aligned} \tag{15}$$

where

σ_{ij}	is the stress tensor
n_i	is the normal vector
t_i	is the tangent vector
A, B	denote media A and media B, respectively

Let medium A be a compressible fluid; the stress tensor then becomes (from Eq. (12)

$$\sigma^{\mathrm{A}}_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)$$

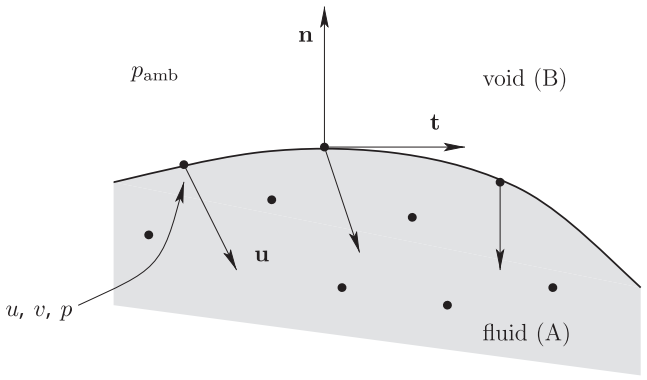

Fig. 1. Sketch of a free surface. Computational nodes carry velocity components u, v, and density ρ and are shown as dots. The surrounding media is considered void with a constant pressure p_{amb}.