Computers & Fluids 84 (2013) 218–230

Contents lists available at SciVerse ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

Calculation of the interface curvature and normal vector with the level-set method

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ARTICLE INFO

Article history:
Received 20 January 2012
Received in revised form 9 April 2013
Accepted 10 June 2013
Available online 18 June 2013

Keywords:
Level-set method
Curvature discretization
Normal-vector discretization
Curve-fitting discretization scheme
Finite differences
Ghost-fluid method

ABSTRACT

This article addresses the use of the level-set method for capturing the interface between two fluids. One of the advantages of the level-set method is that the curvature and the normal vector of the interface can be readily calculated from the level-set function. However, in cases where the level-set method is used to capture topological changes, the standard discretization techniques for the curvature and the normal vector do not work properly. This is because they are affected by the discontinuities of the signed-distance function half-way between two interfaces. This article addresses the calculation of normal vectors and curvatures with the level-set method for such cases. It presents a discretization scheme based on the geometry-aware curvature discretization by Macklin and Lowengrub [1]. As the present scheme is independent of the ghost-fluid method, it becomes more generally applicable, and it can be implemented into an existing level-set code more easily than Macklin and Lowengrub's scheme [1]. The present scheme is compared with the second-order central-difference scheme and with Macklin and Lowengrub's scheme [1], first for a case with no flow, then for a case where two drops collide in a 2D shear flow, and finally for a case where two drops collide in an axisymmetric flow. In the latter two cases, the Navier–Stokes equations for incompressible two-phase flow are solved. The article also gives a comparison of the calculation of normal vectors with the direction difference scheme presented by Macklin and Lowengrub in [2] and with the present discretization scheme. The results show that the present discretization scheme yields more robust calculations of the curvature than the second-order central difference scheme in areas where topological changes are imminent. The present scheme compares well to Macklin and Lowengrub's method [1]. The results also demonstrate that the direction difference scheme [2] is not always sufficient to accurately calculate the normal vectors.

1. Introduction

The level-set method was introduced by Osher and Sethian [3]. It is designed to implicitly track moving interfaces through an isocontour of a function defined in the entire domain. In particular, it is designed for problems in multiple spatial dimensions in which the topology of the evolving interface changes during the course of events, cf. [4].

This article addresses the calculation of interface geometries with the level-set method. This method allows us to calculate the normal vector and the curvature of an interface directly as the first and second derivatives of the level-set function. These calculations are typically done with standard finite-difference methods. Since the level-set function is chosen to be a signed-distance function, in most cases it will have areas where it is not smooth. Consider for instance two colliding drops where the interfaces are captured with the level-set method, see Fig. 1. The derivative of the level-set function will not be defined at the points outside the drops that have an equal distance to both drops. When the drops are in near contact, this discontinuity in the derivative will lead to significant errors when calculating the interface geometries with standard finite-difference methods. For convenience the areas where the derivative of the level-set function is not defined will hereafter be referred to as kinks.

To the authors knowledge, this issue was first described in [2], where the level-set method was used to model tumour growth. Here Macklin and Lowengrub presented a direction difference to treat the discretization across kinks for the normal vector and the curvature. They later presented an improved method where curve fitting was used to calculate the curvatures [1]. This was further expanded to include the normal vectors [5].

An alternative method to avoid the kinks is presented in [6], where a level-set extraction technique is presented. This technique uses an extraction algorithm to reconstruct separate level-set functions for each distinct body.

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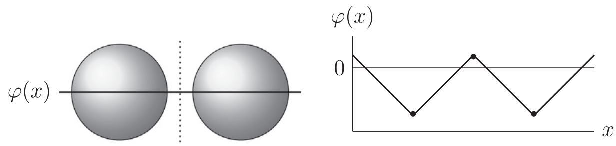
http://dx.doi.org/10.1016/j.compfluid.2013.06.004



(a) Drops in near contact (b) A slice of the level-set function

Fig. 1. (a) Two drops in near contact. The dotted line marks a region where the derivative of the level-set function is not defined. (b) A one-dimensional slice of the level-set function $\varphi(x)$. The dots mark points where the derivative of $\varphi(x)$ is not defined.

Accurate calculation of the curvature is important in many applications, in particular in curvature-driven flows. There are several examples in the literature of methods that improve the accuracy of the curvature calculations, but that do not consider the problem with the discretization across the kinks. The authors in [7] use a coupled level-set and volume-of-fluid method based on a fixed Eulerian grid, and they use a height function to calculate the curvatures. In [8] a refined level-set grid method is used to study two-phase flows on structured and unstructured grids for the flow solver. An interface-projected curvature-evaluation method is presented to achieve converging calculation of the curvature. Marchandise et al. [9] adopt a discontinuous Galerkin method and a pressure-stabilized finite-element method to solve the level-set equation and the Navier–Stokes equations, respectively. They develop a least-squares approach to calculate the normal vector and the curvature accurately, as opposed to using a direct derivation of the level-set function. This method is used by Desjardins et al. [10], where they show impressive results for simulations of turbulent atomization.

This article is a continuation of the work presented in [11]. It applies the level-set method and the ghost-fluid method to incompressible two-phase flow in two dimensions. A curve-fitting discretization scheme is presented which is based on the geometry-aware discretization given in [1]. This scheme is mainly applied to the curvature discretization. The normal vectors are calculated both with the direction difference described in [2] and with a combination of the direction difference and the curve-fitting discretization scheme.

The main advantage of the present scheme compared to the geometry-aware discretization [1] is that it is independent of the ghost-fluid method. That is, in [1] Macklin and Lowengrub calculate the curvature values directly on the interface when it is needed by the ghost-fluid method, whereas we compute the curvature values at the global grid points, indendent of the ghost-fluid method. Because of this, the scheme can be implemented more easily into existing Navier–Stokes codes employing the level-set method, since only small parts of the existing codes need modification. It is also more generally applicable, for instance it can be used with the continuum surface-force method [15]. Further, it allows for more accurate curvature values in models that require curvature values on the grid instead of on the interface, e.g. surfactant models [12–14].

The article starts by briefly describing the governing equations for two-phase flow and the level-set method in Section 2. It continues in Section 3 with a description of the numerical methods that are used for their solution. Then the discretization schemes for the normal vector and the curvature are presented in Section 4, followed by a detailed description of the method for curvature discretization in Section 5. Section 6 gives a convergence test and a comparison of the present discretization scheme with the second-order central difference scheme and Macklin and Lowengrub's scheme [1], first on static interfaces in near contact, then on two drops colliding in a 2D shear flow, and finally on a case where two drops collide in an axisymmetric flow. The section is concluded with a comparison of the direction difference scheme [2] with a combination of the direction difference and the curve-fitting discretization schemes for calculating normal vectors. Finally in Section 7 concluding remarks are made.

2. Governing equations

2.1. Navier–Stokes equations for two-phase flow

Consider a two-phase domain $\Omega = \Omega^+ \cup \Omega^-$, where Ω^+ and Ω^- denote the regions occupied by the respective phases. The domain is divided by an interface $\Gamma = \delta\Omega^+ \cap \delta\Omega^-$ as illustrated in Fig. 2. The governing equations for incompressible and immiscible two-phase flow in the domain Ω with an interface force on the interface Γ can be stated as

$$\nabla \cdot u = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \nabla \cdot (\mu(\nabla u + (\nabla u)^T)) + \rho f_b + \int_\Gamma \sigma\kappa n\,\delta(x - x_I(s))\mathrm{d}s, \tag{2}$$

where u is the velocity vector, p is the pressure, f_b is the specific body force, σ is the coefficient of surface tension, κ is the curvature, n is the normal vector which points to Ω^+, δ is the Dirac Delta function, x_I is a parametrization of the interface, ρ is the density and μ is the viscosity. These equations are often called the Navier–Stokes equations for incompressible two-phase flow.

It is assumed that the density and the viscosity are constant in each phase, but they may be discontinuous across the interface. The interface force and the discontinuities in the density and the viscosity lead to a set of interface conditions,

$$[u] = 0, \tag{3}$$

$$[p] = 2[\mu]n \cdot \nabla u \cdot n + \sigma\kappa, \tag{4}$$

$$[\mu\nabla u] = [\mu]((n \cdot \nabla u \cdot n)n \otimes n + (n \cdot \nabla u \cdot t)n \otimes t - (n \cdot \nabla u \cdot t)t \otimes n + (t \cdot \nabla u \cdot t)t \otimes t), \tag{5}$$

$$[\nabla p] = 0, \tag{6}$$

where t is the tangent vector along the interface, $\otimes$ denotes the dyadic product and $[\cdot]$ denotes the jump across an interface, that is

$$[\mu] \equiv \mu^+ - \mu^-. \tag{7}$$

See [16,17] for more details and a derivation of the interface conditions.

2.2. Level-set method

The interface is captured with the zero level set of the level-set function $\varphi(x, t)$, which is prescribed as a signed-distance function. That is, the interface is given by

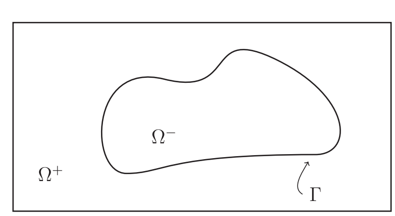


Fig. 2. Illustration of a two-phase domain: The interface Γ separates the two phases, one in Ω^+ and the other in Ω^-.