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High-order methods for decaying two-dimensional homogeneous isotropic turbulence

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ABSTRACT

Numerical schemes used for the integration of complex flow simulations should provide accurate solutions for the long time integrations these flows require. To this end, the performance of various high-order accurate numerical schemes is investigated for direct numerical simulations (DNS) of homogeneous isotropic two-dimensional decaying turbulent flows. The numerical accuracy of compact difference, explicit central difference, Arakawa, and dispersion-relation-preserving schemes are analyzed and compared with the Fourier–Galerkin pseudospectral scheme. In addition, several explicit Runge–Kutta schemes for time integration are investigated. We demonstrate that the centered schemes suffer from spurious Nyquist signals that are generated almost instantaneously and propagate into much of the field when the numerical resolution is insufficient. We further show that the order of the scheme becomes increasingly important for increasing cell Reynolds number. Surprisingly, the sixth-order schemes are found to be in perfect agreement with the pseudospectral method. Considerable reduction in computational time compared to the pseudospectral method is also reported in favor of the finite difference schemes. Among the fourth-order schemes, the compact scheme provides better accuracy than the others for fully resolved computations. The fourth-order Arakawa scheme provides more accurate results for under-resolved computations, however, due to its conservation properties. Our results show that, contrary to conventional wisdom, difference methods demonstrate superior performance in terms of accuracy and efficiency for fully resolved DNS computations of the complex flows considered here. For under-resolved simulations, however, the choice of difference method should be made with care.

1. Introduction

The physics of two-dimensional turbulence have been elucidated substantially during the past decades by theoretical models, intensive numerical investigations, and dedicated soap film experiments [1]. Two-dimensional turbulence research efforts have applicability in geophysics, astronomy and plasma physics, in which numerical experiments play a large role. One of the most important reasons for studying two-dimensional turbulence is to improve our understanding of geophysical flows in the atmosphere and ocean [2–8]. We may also find two-dimensional flows in a wide variety of situations such as flows in rapidly rotating systems and flows in a fluid film on top of the surface of another fluid or a rigid object [9].

Two-dimensional turbulence behaves in a profoundly different way from three-dimensional turbulence due to different energy cascade behavior, and follows the Kraichnan–Batchelor–Leith (KBL) theory [10–12]. In three-dimensional turbulence, energy is transferred forward, from large scales to smaller scales, via vortex stretching. In two dimensions that mechanism is absent, and it turns out that under most forcing and dissipation conditions energy will be transferred from smaller scales to larger scales. This is largely because of another quadratic invariant, the potential enstrophy, defined as the integral of the square of the potential vorticity. Despite the apparent simplicity in dealing with two rather than three spatial dimensions, two-dimensional turbulence is possibly richer in its dynamics than three-dimensional turbulence due to its conservation properties, such as its inverse energy and forward enstrophy cascading mechanisms. Danilov and Gurarie [13] and Tabeling [14] reviewed both theoretical and experimental two-dimensional turbulence studies along with extensions into geophysical flow settings. More recent reviews on two-dimensional turbulence are also provided by Clercx and van Heijst [15] and Boffetta and Ecke [16]. Recent studies in two-dimensional turbulence, both forced (stationary) turbulence [17–20] and unforced (decaying) turbulence [21–23] provide high resolution computational confirmation of the KBL theory.

Simulation of turbulent and other convection-dominated unsteady flows using direct numerical simulation (DNS) requires a

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numerical method that properly resolves all the multiscale flow structures [24]. Since high accuracy is crucial in numerical simulation of complex flows with multiscale structures, such as the unsteady evolution of a turbulent flow field, most two-dimensional turbulence studies have been performed using pseudospectral methods based on fast Fourier transform (FFT) algorithms [21,16]. Simulations performed by the lattice Boltzmann method (LBM) have been also presented for two-dimensional decaying turbulence [25]. Pseudospectral methods are highly accurate but mostly limited to ideal geometries such as rectangular or circular domains. Discretization methods such as finite difference, finite element, or finite volume methods are often preferred in more realistic problems. Finite difference methods offer an attractive alternative to spectral methods in the direct and large eddy simulations (LES) of turbulence providing reasonable accuracy coupled with relative ease of implementation in simple and complex flow geometries [26–29]. Computational algorithms developed in the past were mainly designed for solving large-scale fluid dynamics problems using second-order spatial accuracy [30–32]. These algorithms usually have rather significant dispersion errors and if they are not centered schemes they also have large dissipation errors, making it hard to accurately compute fine structures in the flow field using them [33]. There are two ways to improve the resolution of these methods; one is to refine the grid and the other is to construct a high-order accurate scheme. Our approach here is to test and evaluate different high-order formulations for instantaneous and statistical properties of two-dimensional turbulence and compare their accuracy and efficiency with those of the pseudospectral and the second-order schemes. Furthermore, it has been shown by Kravchenko and Moin [34] that the subgrid-scale models in LES are effective only if central discretization of order higher than two is employed. With this in mind, we will investigate the behavior of four different families of high-order accurate finite difference methods in the decay of two-dimensional isotropic turbulence.

High-order finite difference schemes can be formulated to reduce the truncation errors associated with the difference approximations. A straightforward Taylor series expansion of a pointwise discretization under certain assumptions results in a family of the explicit difference (ED) schemes. The compact difference (CD) schemes feature high-order accuracy with smaller stencils and smaller truncation errors than the ED schemes, and have been employed as an alternative to spectral methods in simulations of turbulence with great flexibility [35]. On the other hand, increasing the stencil size allows us to optimize the weight coefficients in the difference equation. This strategy leads to the dispersion-relation-preserving (DRP) schemes [36], which have been used mostly in acoustics. Another strategy to construct a numerical scheme is based on the conservation properties of the discrete form of the equations. Arakawa [37] suggested that the conservation of energy, enstrophy, and skew-symmetry is sufficient to avoid computational instabilities stemming from non-linear interactions. The conservation and stability properties of the Arakawa scheme were investigated by Lilly [31] by means of spectral analysis along with several first and second-order time integration methods. In the present work, we test several Runge–Kutta methods for time integration, although the primary goal here is to analyze the accuracy of these high-order accurate spatial differencing methods for the long-term evolution of complex two-dimensional turbulent flows. For finite difference schemes, the combination of differentiation errors and non-linear truncation and aliasing errors, which usually manifest themselves in the high wavenumbers of the resolved scales, determines the overall error at the small scales. Looking at the accuracy of the whole solution procedure we also investigate the resolution requirements for these finite difference families, the effects of the order of the schemes, and the importance of the global conservation properties.

The paper is organized as follows: the mathematical formulation of the problem is given in Section 2. The numerical methods are presented in Section 3 with descriptions of high-order accurate spatial discretization schemes, temporal discretization algorithms, and an efficient fast Poisson solver algorithm. These schemes are validated in Section 4 by simulating the Taylor–Green decaying vortex benchmark problem for the unsteady incompressible Navier–Stokes equations. The effective accuracies of these methods are also provided in this section, and are confirmed to be the theoretical accuracies of the schemes. Section 5 presents a careful numerical investigation of their performance for a challenging benchmark problem which consists of strong shear layers. The results for two-dimensional isotropic homogeneous decaying turbulence are provided in Section 6. The behavior of these nine different spatial schemes are tested in terms of accuracy and efficiency. The effects of several explicit Runge–Kutta time advancement techniques on the whole solution procedure are also analyzed. In addition, the Reynolds number (Re) dependency of the turbulence statistics is illustrated in this section. Final conclusions and some comments on the performance of these schemes are drawn in Section 7.

2. Mathematical model

The governing equations for two-dimensional incompressible flows can be written in a dimensionless form of the vorticity-stream function formulation as

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{1}$$

along with the kinematic relationship between vorticity and stream function according to a Poisson equation, which is given as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega. \tag{2}$$

From a computational point of view, this formulation has several advantages over the primitive variable formulation. It eliminates pressure from the Navier–Stokes equations and hence has no corresponding odd–even decoupling between the pressure and velocity components, as well as projection inaccuracies usually observed in fractional step approaches [38]. Therefore, the usage of a collocated grid does not produce any spurious modes in the vorticity-stream function formulation. The vorticity-stream function formulation automatically satisfies the divergence-free condition and allows one to reduce the number of equations to be solved.

The main objective of our work is to test and evaluate different frameworks for high-order accurate finite difference schemes and compare them with a spectrally accurate pseudospectral method for two-dimensional isotropic turbulent flows. In fact, to be able to compare the numerical schemes more precisely we restricted ourselves to periodic boundary conditions and a uniform Cartesian grid. Consequently, we eliminated errors coming from the mesh non-uniformities and inconsistent boundary schemes. It should also be noted that using the vorticity-stream function formulation on a collocated grid provides us with an ideal computational setting in which to test the characteristics of the numerical schemes by eliminating any possible errors coming from projection inaccuracies.

3. Numerical methods

The objective of the present work is to test and evaluate different frameworks for high-order accurate finite difference schemes and compare them with a spectrally accurate pseudospectral