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Eulerian modeling of turbulent dispersion from a point source

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ABSTRACT

We derive a frame-invariant Eulerian model of turbulent dispersion that takes into account the effect of velocity correlation on turbulent diffusivity.

We give a mathematical derivation of the model, we implement it and simulate different benchmark flows. We also simulate the same flows with a simpler Eulerian model, which does not take into account the effect of velocity correlation. We compare the results of both models with analytical and experimental results from the literature and we find that our model gives excellent agreement with the literature data and that it is significantly more accurate than the simpler model.

1. Introduction

Taylor [1] made a Lagrangian analysis of the dispersion of a particle in stationary homogeneous turbulence. He showed that turbulent dispersion varies in time and derived the asymptotic expressions of dispersion for short and large times. The result of Taylor has been widely applied in Lagrangian models (e.g. by Anand and Pope [2]), and in Eulerian models where the turbulent diffusivity D_T is expressed as an explicit function of time (e.g.Deardorff [3]). Frame-invariant formulations of this theory in the Eulerian reference frame are less common. For example Deardorff [3] formulated closures for the second- and third-moment rate equations for diffusion, but found that little or no advantage was to be gained using such closures. Nevertheless, Eulerian modeling of turbulent dispersion is of paramount importance for many applications. We propose therefore a new frame-invariant Eulerian model of turbulent dispersion.

2. Dispersion of a single particle

One of the simplest arrangements to model is the dispersion of a fluid particle released from a point source into a homogeneous stationary turbulent flow. The reference frame has its origin at the point source, the fluid moves with constant average velocity and turbulence is characterized by the r.m.s. velocity u' and by the Lagrangian time scale T_L. This hypothetical experiment yields the following fundamental result first obtained by Taylor [1]:

$$\sigma_X \propto u't \quad \text{for} \quad t << T_L \tag{1}$$

$$\sigma_X \propto \sqrt{2u'^2 T_L t} \quad \text{for} \quad t >> T_L \tag{2}$$

where σ_X is the standard deviation of the position X of the particle and t is the time that has elapsed since the particle was released. If turbulence is isotropic and stationary and the Lagrangian velocity autocorrelation function $\rho(s)$ is known, σ_X can be written as:

$$\sigma_X^2 = 2u'^2 \int_0^t (t-s)\rho(s)ds \tag{3}$$

If f is the probability density function of the position of the fluid particle, and it is assumed that f is Gaussian, the dispersion of the fluid particle can be expressed in terms of a diffusion equation:

$$\frac{\partial f}{\partial t} = D_f(t)\nabla^2 f \tag{4}$$

where the diffusivity $D_f(t)$ is a function of t:

$$D_f(t) = \frac{d}{dt}(\sigma_X^2/2) \tag{5}$$

Eq. (4) can also be interpreted as the dispersion of a pulse of tracer injected at time $t = 0$. In this case f represents the mean concentration of the tracer. Eqs. (3) and (5) give:

$$D_f(t) = u'^2 \int_0^t \rho(s)ds \tag{6}$$

assuming that:

$$\rho(s) = e^{-|s|/T_L} \tag{7}$$

(this expression was shown to approximate well the autocorrelation function in isotropic stationary turbulence (e.g. Sato and Yamamoto[4])) it follows that:

$$D_f(t) = u'^2 T_L (1 - e^{-t/T_L}) \tag{8}$$

Zimont [5,6] noticed that (8) is a solution to the following relaxation equation:

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$$\frac{dD_f}{dt} = \frac{u'^2 T_L - D_f}{T_L} \tag{9}$$

Both Eqs. (8) and (9) represent the evolution of diffusivity experienced by a Lagrangian particle (or by a pulse of tracer) that is released from the point source at time $t = 0$. We refer to the factor between parentheses in Eq. (8) as "the effect of correlation". After a sufficiently large time ($t \approx 3T_L$) the effect of correlation becomes negligible. In fact, in Eulerian models, the effect of correlation is often neglected. This approximation, however, implies a significant error if the relevant time scales are small ($t < 3T_L$).

3. Dispersion of several particles

If N particles are released at different times t_{0i} ($0 < i < N + 1$), each particle will experience a different diffusivity:

$$D_i(t) = u'^2 T_L\{1 - \exp[-(t - t_{0i})/T_L]\} \tag{10}$$

that is the solution to equation:

$$\frac{dD_i}{dt} = \frac{u'^2 T_L - D_i}{T_L} \tag{11}$$

In order to provide a more intuitive representation from an Eulerian point of view, we consider the dispersion of pulses of tracer rather than of fluid particles. At each time t_{0i} a pulse of tracer is injected at the point source. Eq. (10) represents the turbulent diffusivity of the tracer injected at time t_{0i}. Each tracer will disperse according to the diffusion equation (equivalent to Eq. (4)):

$$\frac{Dc_i}{Dt} = D_i(t)\nabla^2 c_i = \nabla^2[D_i(t)c_i] \tag{12}$$

where c_i is the concentration of the tracer injected at time t_{0i}. The diffusivity $D_i(t)$ can be moved inside the Laplacian operator because it is a function of time only. If the same substance is used as a tracer for all N injections we will encounter an apparent paradox: at a given point in the domain, there will be N diffusivities for the same substance. If we sum Eqs. (12), we get:

$$\frac{D\sum_i c_i}{Dt} = \sum_i \nabla^2[c_i D_i(t)] \tag{13}$$

If we call the total concentration of tracer c:

$$c = \sum_i c_i \tag{14}$$

we can rewrite Eq. (13) as:

$$\frac{Dc}{Dt} = \nabla^2(D_c c) \tag{15}$$

where D_c is an average:

$$D_c = \frac{\sum_i D_i(t) c_i}{c} \tag{16}$$

In order to solve Eq. (15) we need to determine D_c. Instead of using Eq. (16), which implies that Eq. (11) is solved for each pulse, we use Eqs. (11) and (12) to build the following transport equation for the product $c_i D_i$:

$$\frac{D(c_i D_i)}{Dt} = c_i \frac{d(D_i)}{dt} + D_i \frac{D(c_i)}{Dt} = c_i \frac{u'^2 T_L - D_i}{T_L} + \nabla^2 D_i^2 c_i \tag{17}$$

summing Eq. (17) for all values of i gives:

$$\frac{D(cD_c)}{Dt} = c\frac{u'^2 T_L - D_c}{T_L} + \nabla^2 \sum_i D_i^2 c_i \tag{18}$$

we are not able to express rigorously the last term in Eq. (18) in terms of c and D_c, therefore, we need to approximate Eq. (18), for example as follows:

$$\frac{D(\Pi)}{Dt} = \frac{cu'^2 T_L - \Pi}{T_L} + \nabla^2(D_c \Pi) \tag{19}$$

where we have defined $\Pi \equiv cD_c$. Unfortunately, we cannot implement Eqs. (15) and (19) in the software that we use due to the unusual form of the diffusion terms. Therefore, we implement the following approximate form of Eqs. (15) and (18):

$$\frac{Dc}{Dt} = \nabla(D_c \nabla c) \tag{20}$$

$$\frac{D\Pi}{Dt} = \frac{cu'^2 T_L - \Pi}{T_L} + \nabla(D_c \nabla \Pi) \tag{21}$$

The peculiarity of Eqs. (18), (19), and (21) is that the first term on the right hand side of the equations grows proportionally to the concentration c. This implies that Π and D_c grow only if the tracer is present. Therefore, in the equation there is an intrinsic "switch", that activates the growth of Π (and therefore of D_c) at the time when the tracer appears in the domain. This "switch" resolves the problem of the frame-invariance, because it removes the need of defining the diffusivity as an explicit function of time. Furthermore, this term has a clear physical sense given by the derivation of Eq. (18). Π can be interpreted as an extensive variable corresponding to the intensive variable D_c.

Similar derivations were used by Ghirelli and Leckner [7] to compute the residence time of the fluid and by Ghirelli [8] to compute the "age of turbulence" in a frame-invariant manner.

In the following we use Eqs. (20) and (21) for predicting the dispersion of tracer in turbulent flows.

4. Considerations on the numerical implementation

The numerical implementation of the system of Eqs. (20) and (21) presents the problem that a division by zero can occur in the calculation since D_c is computed as $D_c = \Pi/c$ and c can be zero within the domain. Additionally, as c tends to zero, the value of D_c can grow to values that seem unrealistic. This effect is presumably due to the use of the approximate form (Eq. (21)) instead of the exact equation (Eq. (18)).

The division by zero can be avoided assigning a value to D_c where $c < \delta$, δ being a sufficiently small number. In these areas, the value of D_c has a limited importance, since c is very small. In our calculations we assigned the diffusivity $D_c = u'^2 T_L$ where $c < \delta$. The solution does not change if δ is reduced further, which confirms the solution does not depend on δ. The excessive values of D_c can be avoided imposing an upper limit:

$$D_c < u'^2 T_L \tag{22}$$

We have noticed that D_c tends to exceed the limit (22) only in regions where c is very small (where it tends to δ) and that the solution does not change using limit (22). The advantage of using limit (22) is faster convergence.

5. Simulation and results

We simulate two hypothetical experiments of a point source in homogeneous turbulence, which have analytical solutions given by the theory of Taylor and by the assumption that $\rho(t)$ is given by Eq. (7). In the first hypothetical experiment a single pulse of tracer is injected at a point in a flow with zero mean velocity. In the second experiment a continuous flow of tracer is injected at a point in a flow with non-zero mean velocity.

We also simulate the experiments of dispersion of a tracer injected continuously from a single source in decaying turbulence performed by Warhaft [9].

For all simulations we use the code Fluent 6.3.26. A second order discretization scheme is used to solve Eqs. (20) and (21). In all