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A note on the stability and boundedness of solutions to non-linear differential systems of second order

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Abstract In this work, we are concerned with the investigation of the qualitative behaviors of certain systems of non-linear differential equations of second order. We make a comparison between applications of the integral test and Lyapunov's function approach on some recent stability and boundedness results in the literature. An example is furnished to illustrate the hypotheses and main results in this paper.

1. Introduction and main results

It is well-known that the investigation of stability and boundedness of solutions plays an important role in qualitative theory and applications of differential equations. Besides, the qualitative behaviors of solutions of differential equations of second order, stability, boundedness, etc., play an important role in many real world phenomena related to the sciences and engineering technique fields. However, we would not like to give here the details of applications done in the literature.

In this paper, motivated by the results and method used in (Kroopnick, 2013, 2014; Tunç, 2010, 2016; Tunç and Tunç, 2015, 2016), we obtain some new results on the stability and boundedness problems of certain systems of non-linear differential equations of second order by means of Lyapunov's direct method. The technique of proofs here involves Lyapunov's function approach instead of the integral test used in (Kroopnick, 2013, 2014; Tunç and Tunç, 2015, 2016). We make a comparison between the applications of these methods on the same problems and their conditions. We also extend and improve some results obtained in these sources from linear case to the non-linear case (see, Kroopnick, 2013, 2014; Tunç and Tunç, 2015, 2016). Our aim is to do a contribution to the topic and literature. The results to be established here may be useful for researchers working on the qualitative theory of differential equations. These are the novelty and originality of this paper.

In Kroopnick (2013) considered the second order scalar linear differential equation of the form

$$x'' + a(t)x = 0. \tag{1}$$

Kroopnick (2013) gave two new and elementary proofs proving the stability of solutions and the boundedness of solutions when $t \to \infty$ for the well-known linear differential equation, Eq. (1), where it is given various constraints on $a(.)$. While the results of (Kroopnick, 2013) are not new, the proofs in there are less complex and quite general.

The results of (Kroopnick, 2013) are the following two theorems, respectively.

Theorem A. (Kroopnick, 2013; Theorem I). Let $a(.) \in C^1[0, \infty)$ such that

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$a(t) > 0$ and $a'(t) \geqslant 0$.

Then all solutions of Eq. (1) are bounded as $t \to \infty$ and the absolute values of the amplitudes form a non-increasing sequence.

Theorem B. (Kroopnick, 2013; Theorem II). Let $a(.)$ be a continuous function on $[0, \infty)$ such that

$$a(t) > 0, \quad a'(t) \geqslant 0 \text{ and } K > a(t) > k > 0,$$

where K and k are some positive constants.

Then all solutions of Eq. (1) are bounded as $t \to \infty$, stable and the absolute values of the amplitudes form a non-increasing sequence.

Remark 1. Benefited from the integral test, Kroopnick (2013) proved both of Theorems A and B.

Later, Tunç and Tunç (2015) considered the following vector linear differential equation of the second order

$$\ddot{X} + a(t)X = P(t), \tag{2}$$

where $X \in \Re^n, t \in \Re^+, \Re^+ = [0, \infty); a(.) : \Re^+ \to \Re$ and $P(.) : \Re^+ \to \Re^n$ are continuous functions.

It follows that Eq. (2) represents the system of real second order linear differential equations like

$$\ddot{x}_i + a(t)x_i = p_i(t), (i = 1, 2, ..., n).$$

This shows that Eq. (1) is a special case of Eq. (2).

Throughout this paper the symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and for any $X \in \Re^n$, we define the scalar quantity $\|X\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ and call it the norm of X, the usual Euclidean norm.

Tunç and Tunç (2015) proved the following two theorems, respectively.

Theorem C. We assume that the following assumptions hold:

Let $a(.) \in C^1[0, \infty)$ such that

$$a(t) > 0, \quad a'(t) \geqslant 0,$$

$$\|P(t)\| \leqslant e(t), q(t) = \frac{e(t)}{a(t)}$$

and

$$q(.) \in L^1[0, \infty) \text{ for all } t \in \Re^+.$$

Then all solutions of Eq. (2) are bounded as $t \to \infty$. If $P(t) \equiv 0$ in Eq. (2), then all solutions of Eq. (2) are bounded as $t \to \infty$ and the absolute values of the amplitudes form a non-increasing sequence.

Theorem D. Let $a(.)$ and $P(.)$ be continuous functions on $[0, \infty)$ such that

$$a(t) > 0, a'(t) \geqslant 0, K > a(t) > k > 0,$$

where K and k are some positive constants, and

$$\|P(t)\| \leqslant e(t), \quad q(t) = \frac{e(t)}{a(t)}, \quad q(.) \in L^1[0, \infty).$$

Then all solutions of Eq. (2) are bounded as $t \to \infty$. If $P(t) \equiv 0$ in Eq. (2), then all solutions of Eq. (2) are stable and the absolute values of the amplitudes form a non-increasing sequence.

Remark 2. By means of the integral test, Tunç and Tunç (2015) proved Theorems C and D.

In this paper, instead of Eqs. (1) and (2), we consider the vector non-linear differential equation of the second order

$$\ddot{X} + a(t)X = P(t, \dot{X}) \tag{3}$$

or its equivalent differential system

$$\dot{X}_1 = X_2,$$

$$\dot{X}_2 = -a(t)X_1 + P(t, X_2), \tag{4}$$

where $X = X_1, X_1 \in \Re^n, t \in \Re^+, \Re^+ = [0, \infty); a(.) : \Re^+ \to \Re$ and $P(.) : \Re^+ \times \Re^n \to \Re^n$ are continuous functions.

The continuity of the functions a and P is a sufficient condition to guarantee the existence of solutions of Eq. (3). In addition, we assume that the function P satisfies a Lipschitz condition with respect to $\dot{X}$. In this case, the uniqueness of solutions of Eq. (3) is guaranteed. In particular, for the existence and uniqueness of solutions and some qualitative properties of solutions, we can refer to (Tunç, 2010, 2017; Song et al., 2011; Wang et al., 2010a,b).

It is obvious that Eq. (3) represents the system of real second order non-linear differential equations like

$$\ddot{x}_i + a(t)x_i = p_i(t, \dot{x}_i), \quad (i = 1, 2, ..., n).$$

This shows that Eqs. (1) and (2) are special cases of Eq. (3). That is, our equation, Eq. (3), includes the equations discussed by Kroopnick (2013) and Tunç and Tunç (2015), Eqs. (1) and (2), respectively.

Our first main result is the following theorem.

Theorem 1. We assume that the following assumptions hold:

Let $a(.), P(.) \in C^1[0, \infty)$ such that

$$a(t) > 0, \quad a'(t) \geqslant 0,$$

and there exists a non-negative and continuous function $e = e(t)$ such that

$$\|P(t, X_2)\| \leqslant e(t)\|X_2\|$$

and

$$e(t) \in L^1[0, \infty) \text{ for all } t \in \Re^+,$$

where $L^1(0, \infty)$ denotes the space of Lebesgue integrable functions.

Then all solutions of Eq. (3) are bounded as $t \to \infty$. If $P(t, X_2) \equiv 0$ in Eq. (3), then all solutions of Eq. (3) are bounded as $t \to \infty$ and the zero solution of Eq. (3) is asymptotically stable.