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REVIEW ARTICLE

# Boundedness and stability in third order nonlinear vector differential equations with multiple deviating arguments

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**Abstract** In this paper, we shall establish sufficient conditions for the uniform asymptotic stability and boundedness of solutions of a certain third order vector nonlinear non-autonomous differential equation with multiple deviating arguments, by using a Lyapunov function as basic tool. In doing so we extend some existing results. Example is given to illustrate our results.



## 1. Introduction

As is well known, differential equations with retarded argument are used to describe many phenomena of physical interest. In the last years, there has been increasing interest in obtaining the sufficient conditions for the stability/instability/ boundedness/ ultimately boundedness etc. The problem of the boundedness and stability of solutions of vector differential equations has been widely studied by many authors, who have provided many techniques especially for delay differential equations, for some related contributions, we refer the reader to Graef et al. (2015a,b), Oudjedi et al. (2014), Remili and Beldjerd (2014), Remili and Oudjedi (2014a,b, 2016a,b,c,d,e), Remili et al. (2016), Tunç (2006a,b), Tunç and Gözen (2014), Zhengxin et al. (2010a,b, 2015), Zhihui and Jinde (2005, 2007). In the following, we provide some background details regarding the study of various classes of third differential equations.

Tunç (2009) studied the stability and boundedness of the following vector differential equation of third order without delay:

$$X''' + \Psi(X')X'' + BX' + cX = P(t), \tag{1.1}$$

for $P \equiv 0$ and $P \neq 0$ respectively.

Later, Omeike and Afuwape (2010) proved the ultimate boundedness of the same equation.

After that, Tunç and Mohammed (2014) established conditions under which all solutions of third order vector differential equation with delay of the form

$$X''' + \Psi(X')X'' + BX'(t-\tau_1) + cX(t-\tau_1) = P(t), \tag{1.2}$$

tend to the zero solution as $t \longrightarrow \infty$ for $P \equiv 0$ and ultimate boundedness for $P \neq 0$.

Recently, Tunç (2017) adapted Tunç and Mohammed (2014) and used a suitable Lyapunov function to establish criteria which guarantee asymptotic stability of solution of nonautonomous delay differential equation of the third order that is bounded together with its derivatives on the real line,

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and boundedness under explicit conditions on the nonlinear terms of the equation

$$X''' + H(X')X'' + G(X'(t-\tau)) + cX(t-\tau) = F(t, X, X', X''). \tag{1.3}$$

This research is concerned with more general third order nonlinear vector multi-delay differential equations of the form

$$[H(X)X'']' + A(t)\Psi(X')X'' + B(t)\sum_{i=1}^{n} G_i(X'(t-r_i(t))) + C(t)\sum_{i=1}^{n} F_i(X(t-r_i(t))) = 0, \tag{1.4}$$

and

$$[H(X)X'']' + A(t)\Psi(X')X'' + B(t)\sum_{i=1}^{n} G_i(X'(t-r_i(t))) + C(t)\sum_{i=1}^{n} F_i(X(t-r_i(t))) = P(t), \tag{1.5}$$

in which $X \in \mathbb{R}^n$, $F_i, G_i : \mathbb{R}^n \to \mathbb{R}^n$, $H, \Psi : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $A, B, C : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $P : \mathbb{R}_+ \to \mathbb{R}^n$ are continuous differentiable functions with $(F_i(0) = G_i(0) = 0)$ and $H$ is twice differentiable, where $0 \leqslant r_i(t) \leqslant \gamma, r_i'(t) \leqslant \omega_i$ , $0 < \omega_i < 1$ for all $i$, $(i = 1, 2, \ldots, n), \omega_i$ and $\gamma$ are some positive constants, $\gamma$ will be determined later, and the primes in (1.4) and (1.5) denote differentiation with respect to $t, t \in \mathbb{R}^+$.

Finally, the continuity of the functions $H, G_i, \Psi, F_i, P, A, B$ and $C$ guarantee the existence of the solution of (1.4) and (1.5). In addition, we assume that the functions $H, G_i, \Psi$ and $F_i$ satisfy a Lipschitz condition with respect to their respective arguments, like $X$ and $X'$. In this case, the uniqueness of solutions of the Eq. (1.5) is guaranteed.

The motivation of the present work comes from papers mentioned above and the references listed in this paper. It should be also noted that the equations studied here is more general than (1.1), (1.2), (1.3) and that considered in Remili and Oudjedi (2016b).

## 2. Preliminaries

The symbol $\langle X, Y\rangle$ corresponding to any pair $X$ and $Y$ in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, that is, $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$, Thus $\langle X, X\rangle = \|X\|^2$.

**Definition 2.1.** We definite the spectral radius $\rho(D)$ of a matrix $D$ by

$$\rho(D) = \max\{\lambda/\lambda \text{ is eigenvalue of} D\}.$$

**Lemma 2.2.** *For any $D \in \mathbb{R}^{n\times n}$, we have the norm $\|D\| = \sqrt{\rho(D^T D)}$. If D is symmetric then*

$$\|D\| = \rho(D).$$

We shall note all the equivalents norms by the same notation $\|X\|$ for $X \in \mathbb{R}^n$ and $\|D\|$ for a matrix $D \in \mathbb{R}^{n\times n}$.

The following results will be basic to the proofs of Theorems.

**Lemma 2.3.** Afuwape (1983), Afuwape and Omeike (2004), Ezeilo and Tejumola (1966), Ezeilo (1967), Ezeilo and Tejumola (1975), Tiryaki (1999)

*Let D be a real symmetric positive definite $n \times n$ matrix, then for any Xin $\mathbb{R}^n$, we have*

$$\delta_D\|X\|^2 \leqslant \langle DX, X\rangle \leqslant \Delta_D\|X\|^2,$$

*where $\delta_D, \Delta_D$ are the least and the greatest eigenvalues of D, respectively.*

**Lemma 2.4.** Afuwape (1983), Afuwape and Omeike (2004), Ezeilo and Tejumola (1966), Ezeilo (1967), Ezeilo and Tejumola (1975), Tiryaki (1999)

*Let $Q, D$ be any two real $n \times n$ commuting matrices, then*

(i) The eigenvalues $\lambda_i(QD)(i = 1, 2\ldots, n)$ of the product matrix $QD$ are all real and satisfy

$$\min_{1\leqslant j,k\leqslant n} \lambda_j(Q)\lambda_k(D) \leqslant \lambda_i(QD) \leqslant \max_{1\leqslant j,k\leqslant n} \lambda_j(Q)\lambda_k(D).$$

(ii) The eigenvalues $\lambda_i(Q + D)(i = 1, 2\ldots, n)$ of the sum of matrices $Q$ and $D$ are all real and satisfy.

$$\left\{\min_{1\leqslant j\leqslant n} \lambda_j(Q) + \min_{1\leqslant k\leqslant n} \lambda_k(D)\right\} \leqslant \lambda_i(Q + D) \leqslant \left\{\max_{1\leqslant j\leqslant n} \lambda_j(Q) + \max_{1\leqslant k\leqslant n} \lambda_k(D)\right\}.$$

**Lemma 2.5.** Ezeilo and Tejumola (1966), Ezeilo (1967), Ezeilo and Tejumola (1975), Tiryaki (1999), Mahmoud and Tunç (2016)

Let $H(X)$ be a continuous vector function with $H(0) = 0$.

1) $\frac{d}{dt}\left(\int_0^1 \langle H(\sigma X), X\rangle d\sigma\right) = \langle H(X), X'\rangle.$
2) $\int_0^1 \langle C(t)H(\sigma X), X\rangle d\sigma = \int_0^1\int_0^1 \sigma[\langle C(t)J_H(\sigma\tau X)X, X\rangle]d\sigma d\tau.$

**Lemma 2.6.** Ezeilo and Tejumola (1966), Ezeilo (1967), Ezeilo and Tejumola (1975), Tiryaki (1999), Mahmoud and Tunç (2016)*Let $H(X)$ be a continuous vector function with $H(0) = 0$.*

1) $\langle H(X), H(X)\rangle = 2\int_0^1\int_0^1 \sigma\langle J_H(\sigma X)J_H(\sigma\tau X)X, X\rangle d\sigma d\tau.$
2) $\langle C(t)H(X), X\rangle = \int_0^1 \langle C(t)J_H(\sigma X)X, X\rangle d\sigma.$

**Lemma 2.7.** *Let $H(X)$ be a continuous vector function and that $H(0) = 0$ then,*

$$\delta_H\|X\|^2 \leqslant \int_0^1 \langle H(\sigma X), X\rangle d\sigma \leqslant \Delta_H\|X\|^2,$$

*where $\delta_H, \Delta_H$ are the least and the greatest eigenvalues of $J_H(X)$ (Jacobian matrix of H), respectively.*

**Definition 2.8.** *We definite the spectral radius $\rho(A)$ of a matrix A by*

$$\rho(A) = \max\{\lambda/\lambda \text{ is eigenvalue of } A\}.$$