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Optimal variational iteration method for nonlinear problems

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Abstract This paper focuses on the study of boundary value problems using well-known He's variational iteration method which is coupled with an auxiliary parameter. Three examples are given to show the efficiency and importance of the proposed algorithm. The reliability and accuracy has been proved by comparing our results with the solution obtained by standard variational iteration method.

1. Introduction

Inokuti et al. (1978) proposed a general Lagrange multiplier method to solve nonlinear problems especially in the field of quantum mechanics. Later on, He (1999, 2000, 2007) modified the method to a new kind of an analytical technique for nonlinear problems and named it as variational iteration method (VIM), which is effectively and easily used to obtain solution of nonlinear equations accurately. For example, Belgacem et al. (2015) and Baskonus et al. (2015) obtain solutions of nonlinear fractional differential equations systems (NFDES) through implementation of VIM and concluded that VIM remains a valuable tool for the treatments of NFDES. Bulut and Baskonus (2009) obtain an exact solution of dispersive equation. Wazwaz (2007a, 2007b, 2007c, 2008) applied the method to nonlinear differential equations and pointed out that VIM is a very effective and reliable analytical tool for solving these equations. Saberi and Tamamgar (2008) concluded that the method is highly reliable for integro-differential equations. Goh et al. (2009) applied the method to hyperchaotic system with great success. Uremen and Yildirim (2009) and Sadighi and Ganji (2007) obtain exact solutions of poisson equation and nonlinear diffusion equations respectively. With the passage of time, several modifications were made in He's VIM, which have further improved the efficiency and accuracy of the iterative algorithm to a tangible level.

Moreover, with the passage of time many analytical techniques are developed to solve nonlinear problems. Liao (1992) came up with a new idea, he developed a nonlinear analytical technique called homotopy analysis method (HAM), which is free from assumption of small parameters and can be used to obtain approximate solution of nonlinear problems. In this method, Liao inserted an auxiliary parameter h, which is used to control the convergence of an approximate solution over the domain of the problem. Liao (2003), further generalized the method so called optimal homotopy analysis method (OHAM) for strongly nonlinear differential equations by inserting multiple parameters, which are used to control the convergence of approximate solutions. The optimal value of auxiliary parameters is obtained by minimizing the absolute residual error, which is a reliable, effective and accurate

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method even for higher order of approximation. Different available studies that used the OHAM to solve various nonlinear equations can be seen in the literature (Xu et al., 2015; Nawaz et al., 2015; Ellahi et al., 2015a, 2015b; Zeeshan et al., 2014, 2016).

In this paper, an auxiliary parameter h is inserted into the correctional functional of VIM for boundary value problems. We consume all of the boundary conditions to establish an integral equation before constructing an iterative algorithm to obtain an approximate solution. Thus we establish a modified iterative algorithm that does not contain undetermined coefficients, whereas most previous iterative methods do incorporate undetermined coefficients. It is observed that the coupling algorithm provides a convenient way to control and adjust the convergence region of approximate solution over the domain of the problem. Three examples are given to explicitly reveal the performance and reliability of the suggested algorithm.

2. Variational iteration method (VIM)

To illustrate the steps of variational iteration method, we consider the following general nonlinear ordinary differential equation.

$$Lf(\xi) + Nf(\xi) + g(\xi) = 0. \tag{1}$$

where L and N are linear and nonlinear operator respectively and $g(\xi)$ illustrates an inhomogeneous term. According to VIM (He, 1999, 2000, 2007; Noor and Mohyud-Din, 2008), we can construct the correction functional as follows

$$f_{n+1}(\xi) = f_n(\xi) + \int_0^{\xi} \lambda(Lf_n(s) + N\tilde{f}_n(s) + g(s))ds, \tag{2}$$

where λ is a Lagrange multiplier (He, 1999, 2000, 2007; Noor and Mohyud-Din, 2008), which can be identified optimally via variational theory, f_n is the nth approximate solution, and $\tilde{f}_n$ is consider as a restricted variation, i.e. $\delta\tilde{f}_n = 0$. After identification of Lagrange multiplier, the successive approximations $f_{n+1}(\xi)$, $n \geqslant 0$, of the solution f can be readily obtained. Consequently, the exact solution will be of the form:

$$f(\xi) = \lim_{n\to\infty} f_n(\xi). \tag{3}$$

3. Optimal variational iteration method (OVIM)

To illustrate the steps of optimal variational iteration method, we consider the following second order nonlinear ordinary differential equation.

$$Lf(\xi) + Nf(\xi) + g(\xi) = 0, \quad a \leqslant \xi \leqslant b, \tag{4}$$

subject to the boundary conditions

$$f(a) = \alpha, \quad f(b) = \beta, \tag{5}$$

where $L = \frac{d^2}{d\xi^2}$ is the linear differential operator, N represents the nonlinear operator and $g(\xi)$ illustrate an inhomogeneous term. According to standard VIM, the correction functional is given as

$$f_{n+1}(\xi) = f_n(\xi) + \int_0^{\xi} \lambda\Big(Lf_n(s) + N\tilde{f}_n(s) + g(s)\Big)ds. \tag{6}$$

Making the correction functional stationary, the Lagrange multiplier is identified as $\lambda = s - \xi$, (Noor and Mohyud-Din, 2008; Xu, 2009), we get the following iterative formula

$$f_{n+1}(\xi) = f_n(\xi) + \int_0^{\xi} (s-\xi)(Lf_n(s) + N\tilde{f}_n(s) + g(s))ds. \tag{7}$$

An unknown auxiliary parameter h can be inserted into the iterative formula (7), for $n = 0$, Eq. (7), becomes

$$f_1(\xi) = f(0) + \xi f'(0) + h\int_0^{\xi} (s-\xi)\Big(Lf_0(s) + N\tilde{f}_0(s) + g(s)\Big)ds. \tag{8}$$

for optimal variational iteration method, we will proceed as follows,

From Eq. (8),

$$f(\xi) = f(0) + \xi f'(0) + h\int_0^{\xi}(s-\xi)(Lf(s) + Nf(s) + g(s))ds. \tag{9}$$

Substituting $\xi = a$ and $\xi = b$ in Eq. (9) and solve for $f(0)$, $f'(0)$ we get

$$f(0) = \alpha - a\left(\frac{\alpha-\beta}{a-b}\right) - h\int_0^a (s-a)(Lf(s)+Nf(s)+g(s))ds + \frac{ah}{a-b}\left(\int_0^a (s-a)(Lf(s)+Nf(s)+g(s))ds - \int_0^b (s-b)(Lf(s)+Nf(s)+g(s))ds\right),$$

$$f'(0) = \frac{\alpha-\beta}{a-b} - \frac{h}{a-b}\left(\int_0^a (s-a)(Lf(s)+Nf(s)+g(s))ds + \int_0^b (s-b)(Lf(s)+Nf(s)+g(s))ds\right).$$

Substituting the value of $f(0)$ and $f'(0)$ in Eq. (9) yields

$$\begin{aligned} f(\xi) = {} & \alpha - a\left(\frac{\alpha-\beta}{a-b}\right) + \xi\left(\frac{\alpha-\beta}{a-b}\right) \\ & + h\int_0^{\xi}(s-\xi)(Lf(s)+Nf(s)+g(s))ds \\ & + \frac{h}{a-b}(a-\xi)\int_0^a (s-a)(Lf(s)+Nf(s)+g(s))ds \\ & + \frac{h}{a-b}(\xi-a)\int_0^b (s-b)(Lf(s)+Nf(s)+g(s))ds \\ & - h\int_0^a (s-a)(Lf(s)+Nf(s)+g(s))ds, \end{aligned} \tag{10}$$

which can be solved by the modified iterative algorithm as

$$f_0(\xi) = \alpha - a\left(\frac{\alpha-\beta}{a-b}\right) + \xi\left(\frac{\alpha-\beta}{a-b}\right),$$

$$\begin{aligned} f_1(\xi,h) = {} & f_0(\xi) + h\int_0^{\xi}(s-\xi)(Lf_0(s)+Nf_0(s)+g(s))ds \\ & + \frac{h}{a-b}(a-\xi)\int_0^a (s-a)(Lf_0(s)+Nf_0(s)+g(s))ds \\ & + \frac{h}{a-b}(\xi-a)\int_0^b (s-b)(Lf_0(s)+Nf_0(s)+g(s))ds \\ & - h\int_0^a (s-a)(Lf_0(s)+Nf_0(s)+g(s))ds \end{aligned}$$

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