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An analytical approximate technique to investigate a finite extensibility nonlinear oscillator

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Abstract In this paper, an analytical approximate technique based on modified harmonic balance method is presented to study about the dynamics of a finite extensibility nonlinear oscillator described by Febbo (2011) and Beléndez et al. (2012). Generally, a second-order approximation is only considered in this paper. In the proposed method, the approximate period of oscillations and the corresponding periodic solutions are determined, which are valid for both range of amplitudes $0 < A \leq 0.9$ and $0.9 < A < 1$ of oscillation. The approximate periods obtained in this paper are compared with numerical result (considered to be exact) and other existing results. Firstly, the results are obtained for the amplitude $0 < A \leq 0.9$ and show that the present method gives high accuracy than other existing results. Moreover, the results are also obtained for the rest of the amplitude $0.9 < A < 1$. The relative error measure in this paper is 0.03% for $A = 0.9$ while the relative errors obtained by Febbo (2011) and Beléndez et al. (2012), were 3.53% and 0.60% respectively. On the other hand, Belendez et al. (2012) did not obtain approximate period for $0.9 < A < 1$. In this article, the approximate periods have been determined in the range of value $0.9 < A < 1$ and they have provided better results than the existing result of Febbo (2011).

1. Introduction

Nonlinear vibrations of oscillation problems are essential tool in physical science, mechanical structures, nonlinear circuits, chemical oscillation and other engineering research. Nonlinear vibrations of oscillation systems are modeled by nonlinear differential equations. It is almost difficult to get exact solutions for such nonlinear differential equations. There are several methods which are used to solve nonlinear differential equations. Among one of the widely used methods is perturbation method (Marion, 1970; Krylov and Bogoliubov, 1947; Bogoliubov and Mitropolskii, 1961; Nayfeh and Mook, 1979) where the nonlinear response is small. The perturbation method is not applied when a small parameter is not present in a nonlinear problem. There are many methods (Amore and Aranda, 2005; Cheung et al., 1991; He, 2002; Khan et al., 2012a; Khan and Mirzabeigy, 2014; Saha and Patra, 2013; Yazdi et al., 2010; Khan et al., 2011; Yildirim et al., 2011a; Yildirim et al., 2011b; Khan and Akbarzade, 2012; Khan et al., 2012b; Akbarzade and Khan, 2012; Yildirim et al., 2012; Khan et al., 2013) which are used to solve strongly nonlinear equations.

The harmonic balance method (HBM) (Belendez et al., 2007; Mickens, 1996, 1984; Wu et al., 2006; Lim et al., 2005; Alam et al., 2007; Hosen et al., 2012) is another technique

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for solving strongly nonlinear equations. When a HBM is applied to the nonlinear equations for higher-order approximation, then a set of difficult nonlinear complex equations appears and it is very difficult to solve these complex equations analytically. In previous article, Hosen et al. (2012) solved such nonlinear algebraic equations easily using a truncation principle. Recently, some authors (Razzak, 2016; Hosen and Chowdhury, 2015; Joubari et al., 2015) have used the harmonic balance method for solving various nonlinear oscillator problems.

Furthermore, homotopy analysis method (HAM) (Xu et al., 2015; Nawaz et al., 2015; Zhong and Liao, 2016; Ellahi et al., 2015a,b; Rashidi et al., 2015; Zeeshan et al., 2014) has been developed for solving strongly nonlinear systems. The HAM is based on the homotopy in topology, which is, however, different from perturbation method. The HAM has been successfully applied for nonlinear systems in science and engineering.

The finite extensibility nonlinear oscillator plays an important role in the theory of macromolecules, particularly in the theory of polymer dynamics (Febbo et al., 2008), DNA dynamics (Hatfield and Quake, 1999), and in the simulation of non-Newtonian fluids (Koplik and Banavar, 2003). It is one of the problems that do not contain small parameter. In the previous published article, Febbo (2011) studied the analytically dynamics of a finite extensibility nonlinear oscillator using two different approaches. One involved a second-order linearized harmonic balance (LHB) method to determine analytical approximations to the period of oscillations and periodic solution. The approximate periods obtained using a LHB method were compared with exact result for amplitude $0 < A \leqslant 0.9$ and they produced a relative error of less than 3.53%. However, for the rest amplitude range $0.9 < A < 1$, the relative error for the approximate periods increase exponentially *i.e.,* the second-order approximation is failed to investigate in such cases and the author mentioned that higher-order perturbation solutions are needed in such cases. The limitation of the article Febbo (2011) is that linearization of HB method is not suitable for solving others nonlinear oscillators. Another previous existing article, Beléndez et al. (2012) also obtained the second-order approximation for $0 < A \leqslant 0.9$ with relative error of less than 0.60% using harmonic balance method without linearization. The author Beléndez et al. (2012) also presented some comments on the behavior of the periodic solutions for amplitudes approaching 1.While the author Beléndez et al. (2012) obtained periodic solutions for amplitudes approaching1, the approximate periods were not determined using the harmonic balance method without linearization. Moreover, the solution procedure of Beléndez et al. (2012) is not easy and straightforward.

In this paper, a modified harmonic method has been used to solve the approximate solution of finite extensibility nonlinear oscillator. The limitations of two articles (Febbo, 2011; Beléndez et al., 2012) have been eliminated in this present paper. The present method is completely different from the previous two methods Febbo (2011) and Beléndez et al. (2012). Besides, the new results obtained in this article are also different from previous results, especially those results obtained by Febbo (2011) and Beléndez et al. (2012). Moreover, the results of this article give definitely better results than other existing results. Furthermore, the solution procedure of the present paper is new and different compared to other published these two articles (Febbo, 2011; Beléndez et al., 2012). The main purpose of present paper is that an analytical approximate technique based on modified harmonic method has been presented to obtain the second-order approximation of finite extensibility nonlinear oscillator for both range of amplitudes, $0 < A \leqslant 0.9$ and $0.9 < A < 1$. Although, the trial solution in this paper is the same with the trial solution of Hosen et al. (2012), but the solution procedure is different from Hosen et al. (2012). The most important significance of this method is that its second-order approximation gives good agreement with the corresponding exact results for the amplitude $0.9 < A < 1$.

2. Application of the modified harmonic balance method

2.1. The solution of finite extensibility oscillator for amplitude, $0 < A \leq 0.9$

The non-dimensional equation of motion governing a finite extensibility nonlinear oscillator is (Febbo, 2011; Beléndez et al., 2012)

$$\ddot{x} + \frac{x}{1 - x^2} = 0, \tag{1}$$

with initial conditions

$$x(0) = A, (\text{with } 0 < A < 1) \text{ and } \dot{x}(0) = 0, \tag{2}$$

where the dot indicates differentiation with respect to t and A denotes the oscillation amplitude.

The second-order approximate periodic solution of Eq. (1) is taken in the following form (Hosen et al., 2012; Razzak, 2016)

$$x(t) = A((1 - u)\cos\phi + u\cos 3\phi), \tag{3}$$

where $\phi = \omega t$, ω is an unknown angular frequency and u is constant which is to be further determined.

Eq. (1) can be written as (Febbo, 2011; Beléndez et al., 2012)

$$(1 - x^2)\ddot{x} + x = 0. \tag{4}$$

Substituting Eq. (3) into the left-side of Eq. (4), we have obtained the following Fourier series expansions:

$$(1 - x^2)\ddot{x} + x = c_1 \cos\varphi + c_3 \cos 3\varphi + \cdots, \tag{5}$$

where

$$c_1 = 4 - 4u + (-4 + 3A^2 + 4u + 2A^2u + 25A^2u^2)\omega^2$$

and $c_3 = 4u + (A^2 - 36u + 19A^2u)\omega^2$.

Substituting Eq. (5) into Eq. (4) and then equating the coefficients of the terms $\cos\phi$ and $\cos 3\phi$ equal to zeros, respectively, we obtain

$$4 - 4u + (-4 + 3A^2 + 4u + 2A^2u + 25A^2u^2)\omega^2 = 0 \tag{6}$$

and

$$4u + (A^2 - 36u + 19A^2u)\omega^2 = 0. \tag{7}$$

Eliminating ω from these two Eqs. (6) and (7), we obtain

$$A^2 - 32u + 15A^2u + 32u^2 - 21A^2u^2 - 25A^2u^3 = 0. \tag{8}$$

Eq. (8) can be written as

$$u = \left(A^2 + 15A^2u + 32u^2 - 21A^2u^2 - 25A^2u^3\right)/32. \tag{9}$$

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