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ORIGINAL ARTICLE

Single and dual solutions of fractional order differential equations based on controlled Picard's method with Simpson rule

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Abstract This paper presents a semi-analytical method for solving fractional differential equations with strong terms like (exp, sin, cos,...). An auxiliary parameter is introduced into the well-known Picard's method and so called controlled Picard's method. The proposed approach is based on a combination of controlled Picard's method with Simpson rule. This approach can cover a wider range of integer and fractional orders differential equations due to the extra auxiliary parameter which enhances the convergence and is suitable for higher order differential equations. The proposed approach can be effectively applied to Bratu's problem in fractional order domain to predict and calculate all branches of problem solutions simultaneously. Also, it is tested on other fractional differential equations like nonlinear fractional order Sine-Gordon equation. The results demonstrate reliability, simplicity and efficiency of the approach developed.

1. Introduction

Fractional calculus and fractional differential equations (FDEs) have been considered as a source of many recent innovations during the last few decades, where the extra fractional-order parameters exhibit more flexibility to interpret many natural phenomena in different fields (Das and Pan, 2012; Monje et al., 2010; Petras, 2011; Podlubny, 1999; Semary et al. 2016).

Many approximations based on semi-analytical and numerical techniques were proposed to solve linear and nonlinear fractional- order differential equations that exist in many physical and engineering problems (Baskonus and Bulut, 2015; Baskonus and Bulut, 2016; Bulut et al. 2016; Chen et al., 2015, Diethelm et al., 2002; Gencoglu et al. 2017; Hashemi and Baleanu, 2016; Keshavarz et al. 2014; Parvizi et al., 2015). Picard's method introduced by Émile Picard in 1890, is a basic tool for proving the existence of solutions of initial value problems regarding ordinary first order differential

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equations. Recently, Picard's method was used to analyze and solve the integral and differential equations with different definitions of the derivative (Azarnavid et al., 2015; El-Sayed et al., 2014; Micula, 2015; Rontó et al. 2015; Salahshour et al., 2015; Vazquez-Leal et al., 2015). However, this method cannot provide us with a simple way to adjust and control the convergence region and the rate of giving an approximate series.

In this paper, we construct Picard's method with an auxiliary parameter h which proves very effective in controlling the convergence region of an approximate solution. Also, the combination between Picard's method with an auxiliary parameter and Simpson rule is proposed to solve nonlinear fractional differential equations in the form:

$$D^\beta u(t) + N[u(t)] = 0.\ u^{(i)}(0) = b_i,\quad i : 0(1)n-1, n-1 < \beta \leqslant n, \tag{1}$$

where $N[u(t)]$ contains strong nonlinear terms like (exp, sin, cos,...). The fractional order derivative (D^β) in Caputo sense defined by (Podlubny, 1999):

$$D^\beta u(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-s)^{n-\beta-1}u^{(n)}(s)ds,\quad n-1<\beta<n, \tag{2}$$

and (J^β) is the Riemann–Liouville fractional integral operator of order $\beta \geqslant 0$ and defined by:

$$J^\beta u(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)ds. \tag{3}$$

The important property of the Caputo fractional derivative is:

$$J^\beta D^\beta u(t) = u(t) - \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!},\quad n-1<\beta\leqslant n. \tag{4}$$

2. The methodology

We apply the Riemann–Liouville integral of order β (J^β) on Eq. (1) and after making use of the property (4), we get the integrated form of Eq. (1), namely

$$u(t) = \sum_{k=0}^{n-1} u^{(k)}(0)\frac{t^k}{k!} - J^\beta N[u(t)], \tag{5}$$

where $u^{(k)}(0) = b_k, k = 0, 1, .., n-1$. Applying Picard's method to the integral Eq. (5), the solution can be reconstructed as follows:

$$u_{m+1}(t) = \sum_{k=0}^{n-1} b_k\frac{t^k}{k!} - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}N[u_m(s)]ds,\quad m\geqslant 0. \tag{6}$$

Adding and subtracting the term $\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}D^\beta u_m(s)ds$ in the right-hand side of (6), the iterative formula (6) becomes:

$$u_{m+1}(t) = \sum_{k=0}^{n-1} b_k\frac{t^k}{k!} - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{D^\beta u_m(s) + N[u_m(s)]\}ds + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}D^\beta u_m(s)ds. \tag{7}$$

Using Caputo fractional order derivative (4), then Eq. (7) becomes:

$$u_{m+1}(t) = \sum_{k=0}^{n-1} b_k\frac{t^k}{k!} - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{D^\beta u_m(s) + N[u_m(s)]\}ds + u_m(t) - \sum_{k=0}^{n-1} u_m^{(k)}\frac{t^k}{k!}. \tag{8}$$

The successive approximation $u_m(t)$ must satisfy the initial conditions, for that $u_m^{(k)} = b_k$ and the iterative formula (8) becomes:

$$u_{m+1}(t) = u_m(t) - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\{D^\beta u_m(s) + N[u_m(s)]\}ds. \tag{9}$$

The property (4) is right for the integer order case and one can prove it easily using integration by parts. So, it should be emphasized that the iteration formula (9) is suitable to solve the problem (1) for integer and fractional orders. The variational iteration method (VIM) is one of the famous techniques used to solve linear and nonlinear differential equations (Ghaneai and Hosseini, 2015; He, 1999; Wazwaz, 2009a; Semary and Hassan, 2015). To solve the integer order differential Eq. (1) by the variational iteration method (He, 1999; Wazwaz, 2009a), one can construct an iteration formula for the problem (1) as follows:

$$u_{m+1} = u_m + \int_0^t \lambda(s)(D^n u_m(s) + N[u_m(s)])ds, \tag{10}$$

where $\lambda(s)$ is a general Lagrange multiplier and it is equal $-\frac{(t-s)^{n-1}}{n-1!}$ (Wazwaz, 2009a).

Remark: *The Picard iterative formula (9) is the same variational iterative formula generated by the variational iteration method (10) when $\beta = n$ and the general Lagrange multiplier*

$$\lambda(s) = -\frac{(t-s)^{n-1}}{\Gamma(n)}.$$

2.1. Controlled Picard's method with Simpson rule

We consider the nonlinear fractional order differential Equation (1) in the form:

$$F[t, u(t), \beta] = D^\beta u(t) + N[u(t)] = 0. \tag{11}$$

Multiply h to both sides in Eq. (11) to become:

$$hF[t, u(t), \beta] = 0, \tag{12}$$

where h is an auxiliary parameter. Adding and subtracting $D^\beta u(t)$ from the left-hand side of (12) to become in the form:

$$D^\beta u(t) + hF[t, u(t), \beta] - D^\beta u(t) = 0, \tag{13}$$

and setting $N_1(t, u) = hF[t, u(t), \beta] - D^\beta u(t)$, the Eq. (13) is given by:

$$D^\beta u(t) + N_1(t, u) = 0. \tag{14}$$

Appling the Picard iteration Eq. (9) to the equation (14), we get:

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