Journal of the Association of Arab Universities for Basic and Applied Sciences (2016) **xxx**, xxx–xxx

University of Bahrain

Journal of the Association of Arab Universities for Basic and Applied Sciences

www.elsevier.com/locate/jaaubas
www.sciencedirect.com

Stability and ultimate boundedness of solutions of some third order differential equations with delay

M. Remili *, **D. Beldjerd**

University of Oran 1 Ahmed Ben Bella, Department of Mathematics, 31000 Oran, Algeria

Received 5 February 2016; revised 24 March 2016; accepted 2 May 2016

KEYWORDS

Stability;
Boundedness;
Ultimate boundedness;
Lyapunov functional;
Third order;
Delay equations

Abstract This paper is devoted to study the boundedness, ultimate boundedness, and the asymptotic stability of solutions for a certain class of third-order nonlinear differential equations using Lyapunov's second method. Our results improve and form a complement to some earlier results in the literature.

1. Introduction

The investigation of the qualitative properties of third order differential equations (with and without delay) have been intensively discussed and are still being investigated in the literature. By employing the Lyapunov's method, many good and interesting results have been obtained concerning the boundedness, ultimate boundedness and the asymptotic stability of solutions for certain nonlinear differential equations. See, the papers of Ademola and Arawomo (2013); Ademola et al. (2013); Burton (2005); Hara (1971); Bao and Cao (2009); Pan and Cao (2010, 2011, 2012); Omeike (2010); Oudjedi et al. (2014); Remili and Beldjerd (2014); Remili and Oudjedi (2014); Remili and Damerdji Oudjedi (2014); Li and Lizhi (1987); Tunç (2007a, b, 2010); Yoshizawa (1966) and their references.

In 1992, (Zhu, 1992), established some sufficient conditions to ensure the stability, boundedness and ultimate boundedness of the solutions of the following third order non-linear delay differential equation

$$x''' + ax'' + bx' + f(x(t-r)) = e(t).$$

Recently, (Graef et al., 2015), studied the following third order non autonomous differential equation with delay

$$[g(x(t))x'(t)]'' + (h(x(t))x'(t))' + \varphi(x(t))x'(t) + f(x(t-r)) = e(t),$$

which is more general than those considered by Zhu (1992). Simulated by the above reasons, we investigate the boundedness, ultimate boundedness, and the asymptotic stability of solutions for a kind of third-order differential equation with delay as follows

$$[g(x''(t))x''(t)]' + (h(x'(t))x'(t))' + (\varphi(x(t))x(t))' + f(x(t-r)) = e(t), \tag{1.1}$$

where $r > 0$ is a fixed delay and $e,\ f,\ g,\ h$, and φ are continuous functions in their respective arguments with $f(0) = 0$. The continuity of functions $e,\ f,\ g,\ h$, and φ guarantees the existence of the solution of Eq. (1.1). In addition, it is also supposed that the derivatives $f'(x),\ \ g'(u),\ h'(y)$ and $\varphi'(x)$ exist and are continuous.

The main purpose of this paper is to establish criteria for the uniform asymptotic stability and, uniform ultimate boundedness, of solutions for the third order non-linear differential Eqs. (1.1). The results obtained in this investigation provide

* Corresponding author.
E-mail address: remilimous@gmail.com (M. Remili).

Peer review under responsibility of University of Bahrain.

http://dx.doi.org/10.1016/j.jaubas.2016.05.002
1815-3852

a good supplement to the existing results on the third order nonlinear delay differential equations in the literature as (Zhu, 1992; Graef et al., 2015).

The remainder of this paper is organized as follows. In Section 2, we give a theorem, which deals with asymptotic stability of the zero solution of the delay differential Eq. (1.1) with $e(t) = 0$. In Section 3, we introduced theorem which discusses the uniform boundedness, and uniform ultimate boundedness of the solutions of Eq. (1.1) for the case $e(t) \neq 0$. Eventually, some conclusions are given in Section 4.

2. Stability

Take general nonlinear non-autonomous delay differential equation in the form

$$x' = f(x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \leqslant \theta \leqslant 0, \ t \geqslant 0, \tag{2.1}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r, 0], \ \mathbb{R}^n) : \|\phi\| \leqslant H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Lemma 2.1 Krasovskii, 1963. *If there is a continuous functional $V(t, \phi) : [0, +\infty) \times C_H \to [0, +\infty)$ locally Lipschitz in ϕ and wedges W_i such that*:

(i) *If $W_1(\|\phi\|) \leqslant V(t, \phi)$, $V(t, 0) = 0$ and $V'_{(2,1)}(t, \phi) \leqslant 0$. Then, the zero solution of (2.1) is stable. If in addition $V(t, \phi) \leqslant W_2(\|\phi\|)$ Then, the zero solution of (2.1) is uniformly stable.*

(ii) *If $W_1(\|\phi\|) \leqslant V(t, \phi) \leqslant W_2(\|\phi\|)$ and $V'_{(2,1)}(t, \phi) \leqslant -W_3(\|\phi\|)$. Then, the zero solution*of (2.1) is uniformly asymptotically stable.

Now, suppose that there are positive constants $g_0, g_1, h_0, h_1, \varphi_0, \varphi_1, \delta_0, \delta_1$ and μ_1 such that the following conditions which will be used on the functions that appeared in Eq. (1.1) are satisfied:

(i) $0 < g_0 \leqslant g(u) \leqslant g_1$, $\quad 0 < h_0 \leqslant h(y) \leqslant h_1$, $0 < \varphi_0 \leqslant \varphi(x) \leqslant \varphi_1$,
(ii) $f(0) = 0, \frac{f(x)}{x} \geqslant \delta_0 > 0 \ (x \neq 0)$, and $|f'(x)| \leqslant \delta_1$ for all x,
(iii) $\frac{\delta_1}{\varphi_0} < \mu_1 < \frac{h_0}{g_1}$,
(iv) $\int_{-\infty}^{+\infty} (|g'(u)| + |h'(u)| + |\varphi'(u)|)du < \infty$.

For ease of exposition throughout this paper we will adopt the following notations

$$P(t) = g(x''(t)), \quad \theta_1(t) = \frac{P'(t)}{P^2(t)},$$
$$\theta_2(t) = h'(x'(t))x''(t) \text{ and } \theta_3(t) = \varphi'(x(t))x'(t). \tag{2.2}$$

and

$$\sigma_1(t) = \min\{x''(0), x''(t)\}, \quad \sigma_2(t) = \max\{x''(0), x''(t)\}, \tag{2.3}$$
$$\rho_1(t) = \min\{x(0), x(t)\}, \quad \rho_2(t) = \max\{x(0), x(t)\},$$
$$\psi_1(t) = \min\{x'(0), x'(t)\}, \quad \psi_2(t) = \max\{x'(0), x'(t)\}.$$

For the case $e(t) \equiv 0$, The stability result of this paper is the following theorem.

Theorem 2.2. *If in addition to the hypotheses (i)–(iv), suppose that the following is also satisfied*

$$r < \min\left\{\frac{2g_0(h_0 - \mu_1 g_1)}{g_1^2 \delta_1}, \frac{2g_0(\mu_1\varphi_0 - \delta_1)}{\delta_1(2\mu_1 g_0 + 1)}\right\},$$

Then every solution of (1.1) is uniformly asymptotically stable.

Proof. Eq. (1.1) is equivalent to the following system

$$\begin{aligned} x' &= y \\ y' &= \frac{z}{P(t)} \\ z' &= -\frac{h(y)}{P(t)}z - \theta_2(t)y - \theta_3(t)x - \varphi(x)y - f(x) \\ &\quad + \int_{t-r}^{t} y(s)f'(x(s))ds, \end{aligned} \tag{2.4}$$

The main tool in the proofs of our results is the continuously differentiable functional $W = W(t, x_t, y_t, z_t)$ defined as

$$W(t, x_t, y_t, z_t) = e^{-\frac{\omega(t)}{\mu}} V_1(t, x_t, y_t, z_t) = e^{-\frac{\omega(t)}{\mu}} V_1, \tag{2.5}$$

where

$$V_1 = \mu_1 F(x) + f(x)y + \frac{\varphi(x)}{2}y^2 + \frac{1}{2P(t)}z^2 + \mu_1 yz + \frac{1}{2}\mu_1 h(y)y^2 + \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds, \tag{2.6}$$

$$\omega(t) = \int_0^t Q(s)ds, \quad \text{and} \quad Q(t) = |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|, \tag{2.7}$$

such that $F(x) = \int_0^x f(u)du$ and $\theta_1, \ \theta_2, \ \theta_3$, are defined as (2.2). μ and λ are some positive constants which will be specified later in the proof. We observe that the above functional V_1 can be rewritten as follows

$$V_1 = \mu_1 F(x) + \frac{\varphi(x)}{2}\left(y + \frac{f(x)}{\varphi(x)}\right)^2 - \frac{f^2(x)}{2\varphi(x)} + \frac{1}{2P(t)}(z + \mu_1 P(t)y)^2 + \frac{\mu_1(h(y) - \mu_1 P(t))}{2}y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds.$$

Considering the conditions (i) and (iii), we derive that

$$\frac{\mu_1(h(y) - \mu_1 P(t))}{2} \geqslant \frac{\mu_1(h_0 - \mu_1 g_1)}{2} > 0.$$

It follows that there exists sufficiently small positive constant δ_2 such that

$$\frac{1}{2P(t)}(z + \mu_1 P(t)y)^2 + \frac{\mu_1(h(y) - \mu_1 P(t))}{2}y^2 \geqslant \delta_2 y^2 + \delta_2 z^2. \tag{2.8}$$

Under the hypotheses (i)–(iii), we have

$$\begin{aligned} \mu_1 F(x) - \frac{f^2(x)}{2\varphi(x)} &\geqslant \mu_1 \int_0^x \left(1 - \frac{f'(u)}{\mu_1\varphi(x)}\right)f(u)du \\ &\geqslant \mu_1 \int_0^x \left(1 - \frac{\delta_1}{\mu_1\varphi_0}\right)f(u)du \\ &\geqslant \delta_3 F(x), \end{aligned}$$

where

$$\delta_3 = \mu_1\left(1 - \frac{\delta_1}{\mu_1\varphi_0}\right) > \mu_1\left(1 - \frac{\mu_1}{\mu_1}\right) = 0.$$

Moreover, assumption (ii) implies

Please cite this article in press as: Remili, M., Beldjerd, D. Stability and ultimate boundedness of solutions of some third order differential equations with delay. Journal of the Association of Arab Universities for Basic and Applied Sciences (2016), http://dx.doi.org/10.1016/j.jaubas.2016.05.002