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Minimal spaces with a mathematical structure

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Abstract This paper will discuss, grill topological space which is not only a space for obtaining a new topology but generalized grill space also gives a new topology. This has been discussed with the help of two operators in minimal spaces.

1. Introduction

The concept of grill is well known in topological spaces. Choquet (see Choquet, 1947) introduced this notion in 1947. After that mathematicians like Thrown (see Thrown, 1973) and Chattopadhyay and Thron (1977) and Chattopadhyay et al. (1983) have developed this study at closure spaces, compact spaces, proximity spaces, uniform spaces and many other spaces. Grill topological space (see Choquet, 1947) is a much more new concept in the literature. It was introduced by Roy and Mukherjee (2007) in 2007. Authors like Al-Omari and Noiri (2011, 2012b, 2013), Hatir and Jafari (2010) and Modak(2013a,b,c,d) have studied this field in detail. They have concentrated their study on two operators and generalized sets on this space and obtained different topologies. Modak has shown that new topology can be made from various types of generalized spaces in Modak (2013b,c).

In this paper we shall define two operators on Alimohammady and Roohi's minimal space (see Alimohammady and Roohi, 2005). We also divide the properties of these two operators into two parts. Again we shall try to obtain a new topology with the help of this minimal space with a grill on the same space. However Roy and Mukherjee (2007) and Al-Omari and Noiri (2012a,b) have considered grill topological space for obtaining a new topology. Actually throughout this paper, we are trying to catch the essential space with a grill on the same space that gives a new topology.

2. Preliminaries

Following are the preliminaries for this paper:

Definition 2.1 (Alimohammady and Roohi, 2005). A family $\mathcal{M} \subseteq \wp(X)$ is said to be minimal structure on X if $\emptyset, X \in \mathcal{M}$.

In this case $(X, \mathcal{M})$ is called a minimal space. Throughout this paper $(X, \mathcal{M})$ means minimal space.

Example 2.2 (Alimohammady and Roohi, 2005). Let (X, τ) be a topological space. Then $\mathcal{M} = \tau$, $SO(X)$ (Levine, 1963), $PO(X)$ (Mashhour et al., 1982) and $\alpha O(X)$ (Njastad, 1965) are the examples of minimal structures on X.

Recall the definition of grill:

Definition 2.3 (Choquet, 1947). A subcollection $\mathcal{G}$ (not containing the empty set) of $\wp(X)$ is called a grill on X if $\mathcal{G}$ satisfies the following conditions:

1. $A \in \mathcal{G}$ and $A \subseteq B$ implies $B \in \mathcal{G}$;

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2. $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Example 2.4. Let X be a nonempty set. Then the filter $\mathcal{F}$ (Thron, 1966) and the grill (Choquet, 1947) do not form a minimal structure on X.

Definition 2.5 (Alimohammady and Roohi, 2005) A set $A \in \wp(X)$ is said to be an $m-$ open set if $A \in \mathcal{M}$. $B \in \wp(X)$ is an $m-$ closed set if $X \setminus B \in \mathcal{M}$. We set

$$m - Int(A) = \cup\{U : U \subseteq A, U \in \mathcal{M}\},$$

$$m - Cl(A) = \cap\{F : A \subseteq F, X \setminus F \in \mathcal{M}\}.$$

The Proposition 2.6 of Alimohammady and Roohi (2005) can be restated by the following:

Proposition 2.6. *Let $(X, \mathcal{M})$ be a minimal space. Then for $A, B \in \wp(X)$,*

1. $m - Int(A) \subseteq A$ *and* $m - Int(A) = A$ *if* A *is an* $m-$ *open set.*
2. $A \subseteq m - Cl(A)$ and $A = m - Cl(A)$ if A is an $m-$ closed set.
3. $m - Int(A) \subseteq m - Int(B)$ *and* $m - Cl(A) \subseteq m - Cl(B)$ *if* $A \subseteq B$.
4. $m - Int(A \cap B) \subseteq (m - Int(A)) \cap (m - Int(B))$ *and* $(m - Int(A)) \cup (m - -Int(B)) \subseteq m - Int(A \cup B)$.
5. $m - Cl(A \cup B) \subseteq (m - Cl(A)) \cup (m - Cl(B))$ *and* $m - Cl(A \cap B) \subseteq (m - Cl(A)) \cap (m - Cl(B))$.
6. $m - Int(m - Int(A)) = m - Int(A)$ *and* $m - Cl(m - Cl(B)) = m - Cl(B)$.
7. $x \in m - Cl(A)$ *if and only if every* $m-$ *open set* U_x *containing* $x, U_x \cap A \neq \emptyset$.
8. $(X \setminus m - Cl(A)) = m - Int(X \setminus A)$ *and* $(X \setminus m - Int(A)) = m - Cl(X \setminus A)$.

Definition 2.7 (Alimohammady and Roohi, 2005). A minimal space $(X, \mathcal{M})$ enjoys the property I if the finite intersection of m – open sets is an m – open set.

Example 2.8. Let X be a nonempty set. Let M be the $m-$ structure (see Al-Omari and Noiri, 2012a) on X. Then the space (X, M) is an example of a minimal space with the property I.

A minimal space $(X, \mathcal{M})$ with grill $\mathcal{G}$ on X is called a grill minimal space and denoted as $(X, \mathcal{M}, \mathcal{G})$.

3. $()^{*\mathcal{M}}$-operator

In this section we obtain a topology from the minimal structure and the grill.

Definition 3.1. Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. A mapping $()^{*\mathcal{M}} : \wp(X) \to \wp(X)$ is defined as follows:

$(A)^{*\mathcal{M}} = (A)^{*\mathcal{M}\mathcal{G}} = \{x \in X : A \cap U \in \mathcal{G}$ for all $U \in \mathcal{M}(x)\}$ for each $A \in \wp(X)$, where $\mathcal{M}(x) = \{U \in \mathcal{M} : x \in U\}$.

The mapping $()^{*\mathcal{M}}$ is called $\mathcal{M}$-local function.

3.1. Properties of $()^{*\mathcal{M}}$-operator

Here we have divided the properties of $()^{*\mathcal{M}}$ into two parts. Some of the properties hold at grill minimal space and other properties hold at minimal space with property I.

Theorem 3.2. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space. Then*

1. $(\emptyset)^{*\mathcal{M}} = \emptyset$.
2. $(A)^{*\mathcal{M}} = \emptyset$, *if* $A \notin \mathcal{G}$.
3. *for* $A, B \in \wp(X)$ *and* $A \subseteq B, (A)^{*\mathcal{M}} \subseteq (B)^{*\mathcal{M}}$.
4. *for* $A \subseteq X, (A)^{*\mathcal{M}} \subseteq m - Cl(A)$.
5. *for* $A \subseteq X, m - Cl[(A)^{*\mathcal{M}}] \subseteq (A)^{*\mathcal{M}}$.
6. *for* $A \subseteq X, (A)^{*\mathcal{M}}$ *is an* $m-$ *closed set.*
7. *for* $A \subseteq X, [(A)^{*\mathcal{M}}]^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}}$.
8. $(A)^{*\mathcal{M}\mathcal{G}} \subseteq (A)^{*\mathcal{M}\mathcal{G}_1}$, *where* $\mathcal{G}_1$ *is a grill on* X *with* $\mathcal{G} \subseteq \mathcal{G}_1$.

Proof.

(1) Obvious from definition of $()^{*\mathcal{M}}$.
(2) Obvious from definition of $()^{*\mathcal{M}}$.
(3) Let $x \in (A)^{*\mathcal{M}}$. Then for all $U \in \mathcal{M}(x), U \cap A \in \mathcal{G}$. Again it is obvious that $U \cap B \in \mathcal{G}$ (from definition of grill). Hence $x \in (B)^{*\mathcal{M}}$.
(4) Let $x \notin m - Cl(A)$, then from Proposition 2.6, there is an U_x such that $U_x \cap A = \phi \notin \mathcal{G}$. Implies that $x \notin (A)^{*\mathcal{M}}$. Hence $(A)^{*\mathcal{M}} \subseteq m - Cl(A)$.
(5) Let $x \in m - Cl[(A)^{*\mathcal{M}}]$ and $U \in \mathcal{M}(x)$, then $U \cap (A)^{*\mathcal{M}} \neq \emptyset$. Let $y \in U \cap (A)^{*\mathcal{M}}$ Then $y \in U$ and $y \in (A)^{*\mathcal{M}}$. Therefore $U \cap A \in \mathcal{G}$, and hence $x \in (A)^{*\mathcal{M}}$. Thus $m - Cl[(A)^{*\mathcal{M}}] \subseteq (A)^{*\mathcal{M}}$.
(6) Proof is obvious from Proposition 2.6 and above Property.
(7) From Property 4, $[(A)^{*\mathcal{M}}]^{*\mathcal{M}} \subseteq m - Cl[(A)^{*\mathcal{M}}]$. Again from Property 5,$m - Cl[(A)^{*\mathcal{M}}] \subseteq (A)^{*\mathcal{M}}$. So, $[(A)^{*\mathcal{M}}]^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}}$.
(8) Obvious from definition grill. □

The Authors, Roy, Muhkerjee, Al-Omari, Noiri, Hatir and Jafiri have considered grill topological space for above theorem. But we have shown that the grill minimal space is the sufficient space for the same.

Second type properties of $()^{*\mathcal{M}}$-operator are:

Theorem 3.3. *Let $(X, \mathcal{M}, \mathcal{G})$ be a grill minimal space and $(X, \mathcal{M})$ enjoys the property I. Then*

1. *for* $A, B \subseteq X, (A \cup B)^{*\mathcal{M}} = (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$.
2. *for* $U \in \mathcal{M}$ *and* $A \subseteq X, U \cap (A)^{*\mathcal{M}} = U \cap (U \cap A)^{*\mathcal{M}}$.
3. *for* $A, B \subseteq X, [(A)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}] = [(A \setminus B)^{*\mathcal{M}} \setminus (B)^{*\mathcal{M}}]$.
4. *for* $A, B \subseteq X$ *with* $B \notin \mathcal{G}, (A \cup B)^{*\mathcal{M}} = (A)^{*\mathcal{M}} = (A \setminus B)^{*\mathcal{M}}$.

Proof.

(1) From Theorem 3.2(3), $(A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}} \subseteq (A \cup B)^{*\mathcal{M}}$. For reverse inclusion, suppose that $x \notin (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$. Then there are $U_1, U_2 \in \mathcal{M}(x)$ such that $U_1 \cap A \notin \mathcal{G}, U_2 \cap B \notin \mathcal{G}$ and hence $(U_1 \cap A) \cup (U_2 \cap B) \notin \mathcal{G}$. Now $U_1 \cap U_2 \in \mathcal{M}(x)$ and $(A \cup B) \cap (U_1 \cap U_2) \subseteq (U_1 \cap A) \cup (U_2 \cap B) \notin \mathcal{G}$, so, $x \notin (A \cup B)^{*\mathcal{M}}$. Therefore $(A \cup B)^{*\mathcal{M}} \subseteq (A)^{*\mathcal{M}} \cup (B)^{*\mathcal{M}}$. Hence the result.
(2) From Theorem 3.2(3), $U \cap (U \cap A)^{*\mathcal{M}} \subseteq U \cap (A)^{*\mathcal{M}}$. For reverse inclusion, suppose $x \in U \cap (A)^{*\mathcal{M}}$ and $V \in \mathcal{M}(x)$. Then $U \cap V \in \mathcal{M}(x)$ and $x \in (A)^{*\mathcal{M}}$, implies $(U \cap V) \cap A \in \mathcal{G}$. So $(U \cap A) \cap V \in \mathcal{G}$. This implies that $x \in (U \cap A)^{*\mathcal{M}}$. Thus $x \in U \cap (U \cap A)^{*\mathcal{M}}$.

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