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A simple harmonic balance method for solving strongly nonlinear oscillators

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Abstract In this paper, a simple harmonic balance method (HBM) is proposed to obtain higher-order approximate periodic solutions of strongly nonlinear oscillator systems having a rational and an irrational force. With the proposed procedure, the approximate frequencies and the corresponding periodic solutions can be easily determined. It gives high accuracy for both small and large amplitudes of oscillations and better result than those obtained by other existing results. The main advantage of the present method is that its simplicity and the second-order approximate solutions almost coincide with the corresponding numerical solutions (considered to be exact). The method is illustrated by examples. The present method is very effective and convenient method for solving strongly nonlinear oscillator systems arising in nonlinear science and engineering.

1. Introduction

Nonlinear oscillation problems are essential tool in physical science, mechanical structures, nonlinear circuits, chemical oscillation and other engineering research. Nonlinear vibrations of oscillation systems are modeled by nonlinear differential equations. It is very difficult to obtain periodic solutions of such nonlinear equations. There are several methods used to solve nonlinear differential equations. Among one of the widely used is perturbation method (Marion, 1970; Krylov and Bogoliubov, 1947; Bogoliubov and Mitropolskii, 1961; Nayfeh and Mook, 1979) whereby the nonlinear response is small. On the other hand, there are many methods (Amore and Aranda, 2005; Cheung et al., 1991; He, 2002) used to solve strongly nonlinear equations. The harmonic balance method (HBM) (Belendez et al., 2007; Mickens, 1996, 1984; Wu et al., 2006; Lim et al., 2005; Alam et al., 2007; Hosen et al., 2012) is another technique for solving strongly nonlinear equations. When a HBM is applied to the nonlinear equations for higher-order approximation, then a set of difficult nonlinear complex equations appear and it is very difficult to analytically solve these complex equations. In a recent article, Hosen et al. (2012) solved such nonlinear algebraic equations easily by using a truncation principle. Recently, many authors (Khan et al., 2011, 2012a,b, 2013a; Khan and Mirzabeigy, 2014; Saha and Patra, 2013; Yazdi et al., 2010; Yildirim et al., 2011a,b, 2012; Khan and Akbarzade, 2012; Akbarzade and Khan, 2012; Akbarzade and Khan, 2013) have studied strongly nonlinear oscillators. Khan et al. (2012a) used a coupling method combining homotopy and variational approach. Other authors (Nayfeh and Mook, 1979; Mickens, 2001; Hu and Tang, 2006; Lim and Wu, 2003) used HBM to solve some strongly nonlinear oscillators. But it is a very laborious procedure to obtain higher-order approximations using those methods (Nayfeh and Mook, 1979; Mickens, 2001; Hu and Tang, 2006; Lim and Wu, 2003). Fesanghary et al. (2009) obtained

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a new analytical approximation by variational iterative method (VIM); but the solution contains many harmonic terms. Many analytical techniques (Hosen et al., 2012; Mickens, 1996, 2001; Lim and Wu, 2003; Tiwari et al., 2005; Ozis and Yildirim, 2007; Ghadimi and Kaliji, 2013; Ganji et al., 2009; Zhao, 2009; Akbarzade and Farshidianfar, 2014; Khan et al., 2013a,b) have been used to solve strongly nonlinear oscillator systems having a rational force (such as Duffing-harmonic oscillator: $\ddot{x}+\frac{x^3}{1+x^2}=0$ etc.) and irrational force (mass attached to a stretched wire: $\ddot{x}+x-\frac{x}{\sqrt{1+x^2}}=0$ etc.). Hosen et al. (2012) solved the Duffing-harmonic oscillator by expanding the term $\frac{x^3}{1+x^2}$ into a polynomial form $\ddot{x}+x^3-x^5+\cdots=0$. But this method (Hosen et al., 2012) is valid for small amplitude of oscillations and it is invalid to investigate the nonlinear oscillator $\ddot{x}+x-\frac{\lambda x}{\sqrt{1+x^2}}=0$. On the contrary, other authors (Fesanghary et al., 2009; Khan et al., 2013a,b) have used different analytical techniques to solve these nonlinear oscillators $\ddot{x}+\frac{x^3}{1+x^2}=0$; $\ddot{x}+x-\frac{\lambda x}{\sqrt{1+x^2}}=0$ etc. without expanding. But their solution procedure for determining higher-order approximations of these nonlinear oscillators is not easy or straightforward and the results (obtained by second order approximation) are not more accurate compared with numerical results.

The purpose of this paper is to apply a simple factor on the strongly nonlinear oscillator systems having a rational and an irrational force and to obtain higher-order approximate frequencies and the corresponding periodic solutions by easily solving the sets of algebraic equations with complex nonlinearities. The trial solution (concern of this paper) is the same as that of Hosen et al. (2012). But the solution procedure is different from that of Hosen et al. (2012). To verify the accuracy of the present method, the two complicated nonlinear oscillators ($\ddot{x}+\frac{x^3}{1+x^2}=0$; $\ddot{x}+x-\frac{\lambda x}{\sqrt{1+x^2}}=0$) are chosen as examples. The method provides better result for both small and large amplitudes of oscillations. The significance of this present method is its simplicity, which not only provides a few harmonic terms, but also gives more accurate measurement than any other exiting solutions.

2. The methods

Let us consider the following general strongly nonlinear oscillator systems having a rational or an irrational force:

$$\ddot{x}+\omega_0^2 x+f(x)=0, \tag{1}$$

with initial conditions

$$x(0)=A, \qquad \dot{x}(0)=0, \tag{2}$$

where over dot denotes the derivatives with respect to t, A denotes the maximum amplitude, $f(x)$ is a nonlinear restoring-force function such that $f(-x)=-f(x)$ and $\omega_0 \geqslant 0$.

The approximate periodic solution of Eq. (1) is taken in the form similar to that of Hosen et al. (2012)

$$x_n(t)=A((1-u_3-u_5\cdots)\cos\varphi+u_3\cos 3\varphi+u_5\cos 5\varphi+\cdots),\quad n=0,1,2,\cdots, \tag{3}$$

where $\varphi=\omega t$, ω is an unknown angular frequency and $u_3, u_5, \cdots$ are constants which are to be further determined.

For the first-order approximation (putting $n=0$ and $u_3=u_5=\cdots=0$ in Eq. (3)), Eq. (3) becomes

$$x_0(t)=A\cos\varphi. \tag{4}$$

Eq. (4) also satisfied Eq. (2).

In this paper, Eq. (1) can be re-written as

$$\frac{\ddot{x}+\omega_0^2x+f(x)}{1+x_0^2}=0. \tag{5}$$

Using Eqs. (3) and (4), we have the left-side the following Fourier series expansions:

$$\frac{\ddot{x}+\omega_0^2x+f(x)}{(1+x_0^2)}=c_1\cos\varphi+c_3\cos 3\varphi+c_5\cos 5\varphi+\cdots,, \tag{6}$$

where

$$c_{2n-1}=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{\ddot{x}+\omega_0^2x+f(x)}{1+x_0^2}\right)\cos(2n-1)\varphi\,d\varphi,\quad n=1,2,3,\cdots,. \tag{7}$$

Substituting Eq. (6) for Eq. (5) and then equating the coefficients of the terms cos φ and cos 3φ, cos 5φ,..., we get a set of nonlinear algebraic equations whose solutions provide the unknown coefficients u_3, u_5, $\cdots$ together with the frequency, ω.

3. Examples

3.1. Example 1

Let us consider a one-dimensional, nonlinear Duffing-harmonic oscillator of the form (Mickens, 2001)

$$\ddot{x}+\frac{x^3}{1+x^2}=0, \tag{8}$$

with initial conditions

$$x(0)=A, \qquad \dot{x}(0)=0. \tag{9}$$

Eq. (1) is an example of a conservative nonlinear oscillatory system having a rational form for the non-dimensional restoring force.

Mickens (2001) rearranged Eq. (8) as

$$(1+x^2)\ddot{x}+x^3=0. \tag{10}$$

Applying the lowest order harmonic balance method to Eq. (10), Mickens (2001) obtained an approximate solution of this oscillator. For the higher-order approximation solutions, a set of complicated algebraic equations are involved and it is very difficult to analytically solve. On the other hand, Fesanghary et al. (2009) obtained higher-order solutions (containing up to ninth harmonic terms) from Eq. (10). To overcome these problems, approximation solutions (containing up to third harmonic terms) have been obtained by applying an easy approach to Eq. (10) which is based on HBM. In this article, nonlinear algebraic equations are solved by truncating higher order terms (followed partially by the principle rule presented in Hosen et al. (2012)).

Consider the second-order approximate periodic solution of Eq. (8) is of the form

$$x_1(t)=A((1-u_3)\cos\varphi+u_3\cos 3\varphi). \tag{11}$$

Therefore, the first-order approximation becomes

$$x_0(t)=A\cos\varphi. \tag{12}$$

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