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Multiple states, topology and bifurcations of natural convection in a cubical cavity

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Abstract

A numerical investigation has been conducted to explore the steady nonlinear low Prandtl number flow/thermal transition in a differentially heated cubic cavity. For small values of Rayleigh number (*Ra*), it is observed that initially there was only one symmetric steady-state solution. When the *Ra* was amplified, the system bifurcates from one fixed-point solution to the two stationary solutions, namely, Mode I and Mode II pitchfork bifurcations. This is due to the symmetric nature existing along the vertical and diagonal planes. The flow structure in the present nonlinear system consists of a pair of asymmetric counter-rotating helical cells in a double helix structure, foliated with invariant helically symmetric surfaces containing the fibre-like fluid particle orbits. Also the evolution of different symmetry-breaking orientations on the transverse and diagonal planes of the cavity was noticed. In the Mode I orientation a symmetric vortex coreline was observed. However, in the Mode II orientation a pair of anti-symmetric vortex corelines was observed. Detailed topological study was made based on the rule of Hunt and the structural stability criteria. Also the simulated results were corroborated with numerical evidence. The existence of the critical *Ra* values was ascertained with the aid of the predicted L_2-error norms, thermal/flow iso-contours and streamlines. The route of Mode I orientation was made of the alternate symmetric and asymmetric flows as *Ra* was augmented.

1. Introduction

In recent years, there has been a growing interest in studying the liquid metal flow in cavities subjected to an external temperature gradient. In the process of solidification of binary alloys and crystal growth in melt fluids, the thermal and concentration buoyancy forces can either aid (or oppose) each other depending on the type of alloy and the process of heating (or/and cooling). When the fluid is heated from below, the flow exhibits a very strong and complex nonlinear behavior which is of primary importance in the solar collector design, passive energy storage, crystal growth and in the micro-manufacturing techniques [1]. For example, in the electronic industry the solid/liquid interface is strongly affected by the heat convection [2]. Stable dynamic solutions of these convection problems are important for practical applications because of their impact on the control of dopant distribution. In the past several decades, the Rayleigh–Bérnard problem had been slightly modified so as to make this classical problem more closely related to the destabilized vertical Bridgman crystal growth system. Intensive theoretical, experimental and numerical studies have been done especially in the infinite horizontal layers and in the relatively shallow cavities.

McFadden and Coriell [3] and Impey et al. [4] have studied the two dimensional (2D) model of the directional solidification configuration for the solutal control. The same problem under the low gravity condition was analyzed by Alexander et al. [5]. The thermal control conditions were used by Larroude et al. [6] to investigate the

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2D Rayleigh–Bénard problem. Computational results presented in these papers were used to analyze the dynamic interaction with the solid/liquid transition. The three dimensional (3D) experimental studies of the problem of current interest have been reported by Bratsun et al. [7]. They showed that an initially symmetric cavity flow could become asymmetric as the Grashof number was increased beyond a critical value. Lan and Yang [8] addressed the oscillatory responses to pulse disturbances in the furnace of a 2D vertical zone-melting system and studied the symmetry-breaking steady-state flows in a simplified 3D vertical zone-melting system [9]. Such a symmetry-breaking phenomenon (or pitchfork bifurcation) in a symmetric physical domain is termed as the Coanda effect [10]. According to Bennacer et al. [11] this symmetry-breaking phenomenon has been influenced by the cavity height and the externally applied temperature gradient. Bifurcation in a nonlinear system is a qualitative change in the dynamics of that system. Bennacer et al. [11] also showed that the bifurcation type might change when a controlling parameter (Rayleigh number (Ra) in the current study) was varied. Transition to chaos is also shown to depend on the critical Rayleigh number. For this cavity problem, Davis [12] and Stork and Muller [13] have concluded that the critical Rayleigh number ($Ra = 1708$) was independent of the Prandtl number (Pr). Erenburg et al. [14] studied the solution multiplicity, stability and bifurcation of low Prandtl number steady natural convection in a two-dimensional rectangular cavity. They observed that laminar oscillatory flows exist around each unstable steady-state branch, which leads to a multiplicity of steady and oscillatory state. From the stability diagram they also observed the marginal stability curves corresponding to the steady symmetry-breaking bifurcations.

Recently 3D computations have been limited to high Pr and provided us a general understanding of the flow development [15,16]. Hence, it is important to analyze the inherent nonlinear transition from symmetric to asymmetric equilibrium states with the increased Ra at the low Prandtl number. For the hydrodynamic/thermal instability problems, the direct numerical simulation (DNS) of 3D time-dependent Navier–Stokes equations is extremely time-consuming. The commonly used linear theory can be applied only for the system state not far from equilibrium, *i.e.*, linear theory is appropriate only for the first bifurcation [17]. The experimental evidence of the nonlinear behaviors in a system and measurements of the significant data such as critical values of the governing parameters for the bifurcations are usually of considerable uncertainty and sometimes difficult to obtain. To explore the nonlinear dynamics including the bifurcation and the routes to chaos of the system we have focused our attention on the natural convection in a cubical cavity subjected to the low Pr based on the DNS results. Our aim is to provide a detailed nonlinear analysis of the symmetry-breaking thermal/flow fields in the investigated buoyancy driven cubical cavity. The major bifurcation type seen frequently in the nonlinear system is the pitchfork bifurcation, which is characterized by the appearance of symmetry-breaking solution. This pitchfork bifurcation can produce some unsteadiness in the flow and consequently perturb the solid/liquid interface and the dopant distribution. The onset of pitchfork bifurcation points in the parameter space is, thus, important to be identified.

The paper is organized as follows. The equations and the prescribed boundary conditions, the symmetries of the problem and their implications along with the theoretical details are summarized in Section 2. The subsequent Section 3 deals with the employed numerical methods along with the grid validation. In Section 4, the flow features are extracted and well represented in terms of topology, which is spanned by different kinds of critical points. The critical point analysis was also presented to show that the simulated velocity vector field satisfies the topological rule of Hunt [18] and the structural stability criteria. This is followed by providing the corroborative numerical evidence. Further, the multiple solutions and the symmetry-breaking (or pitchfork bifurcation) phenomenon were analyzed with the help of the iso-contours of heat flux and the flow variables in the wide range of Rayleigh numbers. The vortical nature was analyzed for the two possible solutions existing in the currently investigated nonlinear system. In addition, the global pitchfork bifurcation scenario for the critical parameter Ra was presented. The conclusions are drawn finally in Section 5.

2. Numerical model

The geometry of an axisymmetric cubical cavity (length $L = 1$ m) containing the liquid metal Tin (Sn) of low Pr (=0.01) is shown in Fig. 1. The Cartesian coordinate frame of reference (x, y, z) for this study was chosen as shown in Fig. 1. The bottom wall ($x = 0$) and the left ($y = 0$) and right ($y = 1$ m) vertical side walls ($0 < x < h$, ($h = 0.75$ m)) were maintained at a high temperature (T_H), which corresponds to the temperature of the furnace in a real crystal growth situation. The top wall was maintained at a lower temperature (T_C), which corresponds to the temperature at the solidification front, and the remaining vertical side walls ($h < x < 1$ m) were imposed with the zero heat flux boundary condition. The remaining lateral walls ($z = 0$, 1 m) of the cavity were also assumed to be thermally insulated (adiabatic). The thermal properties of this melt fluid were treated as constants. The above mentioned thermal boundary conditions can be used to model the synthetic production of single crystals, where the crystal grows slowly from a fluid nutrient contained in a crucible of variable geometry. The fluid under current investigation was incompressible and Newtonian along with the Boussinesq approximation made in the equations of motion along the gravity direction.

With the said assumptions, the governing equations to be solved are given below in the specified gravity field $g = (-g, 0, 0)$.