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New weight functions and second order approximation of the Oore–Burns integral for elliptical cracks subject to arbitrary normal stress field

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ABSTRACT

In this paper, by means of a specific coordinate transformation, the singularity of the weight function is overcome. A strong advantage is obtained for a penny-shaped crack. In this case, a new exact formulation is given and a new alternative non-singular integral is proposed in terms of trigonometric functions. The new approach gives a remarkable streamlining of the Galin's function with the advantage of reducing the complexity of the double integral. Furthermore, we give a second order analytical approximation of Oore–Burns integral for elliptical cracks with respect to deviation from the disk. This approach drastically simplifies the computational procedure without loss of accuracy.

1. Introduction

The weight functions technique introduced by Bueckner [7] and Rice [23] has been a crucial development, especially for cracks in a two dimensional body. The Stress Intensity Factors (SIFs) of planar cracks by means of weighted integrals can be calculated by means of exact or approximated equations proposed for many different geometries [11,29]. If the weight function is unknown, accurate results can be obtained by generalising the weight function derived from the displacement function of Petroski and Achenbach, as suggested by Glinka and Shen [12]. They consider the first three terms of the Taylor expansion of the weight function proposed by Sha and Yang [26].

However, in many cases, for a more realistic simulation, cracks should be considered as planar two-dimensional defects in a three-dimensional body [30,10,2,21,24,16–18,25]. Usually the analytic expressions are involute and a numerical approach is hampered by the singular behaviour of the weight function. In many cases the weight function proposed in literature, is obtained from FE results of elliptic or semi-elliptic defects [8,31,5].

For a disk, the exact mode I weight function is known and was proposed by Galin [29]. This equation is formulated in polar coordinates with the origin in the centre of the disk. Although this equation has an explicit analytic form, the integral of the weight function is not easy because of the singular nature of the weight function. Alternative ways were considered in literature in order to obtain the mode I stress intensity factor. In fact, Smith et al. [28], by means of the stress function, gave the mode I stress intensity factor for a penny shape crack under quadratic or cubic expression of the nominal load [27].

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Nomenclature

δ	size of mesh over crack
Ω	crack shape
$\partial\Omega$	crack border
Q	point of Ω
Q'	point of crack border
Δ	distance between Q and $\partial\Omega$
K_I	mode I stress intensity factor
K_{I0}	mode I stress intensity factor in a circle
K_{Irw}	mode I stress intensity factor from Irwin's equation
K_{I1}	Taylor expansion up to first order of K_I for an ellipse
K_{I2}	Taylor expansion up to second order of K_I for an ellipse
ΔK_I	first order term of mode I stress intensity factor for an ellipse
ΔK_{II}	second order term of mode I stress intensity factor for an ellipse
K_{mn}	dimensionless stress intensity factor
$\bar{x}, \bar{y}$	actual Cartesian coordinate system
x, y	dimensionless Cartesian coordinate system
u, v	auxiliary dimensionless coordinate system
p, p_{mn}	reference constants (pressure)
$\bar{a}, \bar{b}$	actual semi-axis of an elliptical crack
a, b	dimensionless semi-axis of an elliptical crack
e	eccentricity of ellipse
$K(e)$	elliptical integral of first kind
$E(e)$	elliptical integral of second kind
σ_n	nominal tensile stress in $\bar{x}, \bar{y}$ actual Cartesian coordinate system
σ	nominal tensile stress in x, y dimensionless Cartesian coordinate system

A general weight function for three-dimensional cracks is known in the literature as the O-integral, given by Oore and Burns [22]. In the ASM Handbook [4], the O-integral was recognised to be a general formally simple expression for SIF, which is suitable for any shape of the embedded crack.

In the case of circular or tunnel cracks, the O-integral perfectly agrees with the well known results of the literature. In order to strongly simplify the assessment of stress intensity factors, the designer often approximates defects with elliptical cracks in complex structures (see for example [13] or [6] in the case of welded joints).

For elliptical cracks, the agreement between O-integral and known results is acceptably good (in the sense of a maximum errors of a few percent) in the range *major axis/minor axis* $\leqslant 2.5$ [18].

The complexity of the evaluation of the O-integral suggested us to simplify the equation in order to obtain some powerful alternative formulation in closed analytical form.

In the case of general crack shape, the derivation of the first order approximation of the Oore–Burns integral is not immediate and it is heavily based on complex analysis, in particular on the residue theorem [17]. Expression of the Oore–Burns integral accurate to the first order in the deviation from a circle, in terms of Fourier series could be useful for a quickly evaluation when the shape of crack moves away from the circular shape. From a mathematical point of view, the key to derive the Taylor expansion of the OB is complex analysis and especially the residue theorem.

In some recent works (see Livieri and Segala [17]), for the O-integral, the authors gave the expression of the first order deviation from the circle for uniform pressures. In this paper we derive a careful closed-form representation of the Oore–Burns integral (hereinafter, OB integral) along elliptic cracks under general pressure. More precisely, we obtain the closed expression of the second order Taylor expansion of the stress intensity factors proposed by Oore and Burns [22] with respect to deviation of the ellipse from the disk. The deviation of an ellipse from the disk is quantitatively described by the parameter $\varepsilon = 1 - b/a$, where a and b are the major and minor semi-axis, respectively. Furthermore, the paper presents a new approach for streamlining of the Galin's function with the advantage of reducing the complexity of the double integral giving a new way for the assessments of many analytical formulae.

2. Basic definitions

Let Ω be an open bounded simply connected subset of the plane as in Fig. 1. We define:

$$f(Q) = \int_{\partial\Omega} \frac{ds}{|Q - P(s)|^2} \tag{1}$$