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Association schemes perspective of microbubble cluster in ultrasonic fields

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ABSTRACT

Dynamics of a cluster of chaotic oscillators on a network are studied using coupled maps. By introducing the association schemes, we obtain coupling strength in the adjacency matrices form, which satisfies Markov matrices property. We remark that in general, the stability region of the cluster of oscillators at the synchronization state is characterized by Lyapunov exponent which can be defined based on the *N*-coupled map. As a detailed physical example, dynamics of microbubble cluster in an ultrasonic field are studied using coupled maps. Microbubble cluster dynamics have an indicative highly active nonlinear phenomenon, were not easy to be explained. In this paper, a cluster of microbubbles with a thin elastic shell based on the modified Keller-Herring equation in an ultrasonic field is demonstrated in the framework of the globally coupled map. On the other hand, a relation between the microbubble elements is replaced by a relation between the vertices. Based on this method, the stability region of microbubbles pulsations at complete synchronization state has been obtained analytically. In this way, distances between microbubbles as coupling strength play the crucial role. In the stability region, we thus observe that the problem of study of dynamics of *N*-microbubble oscillators reduce to that of a single microbubble. Therefore, the important parameters of the isolated microbubble such as applied pressure, driving frequency and the initial radius have effective behavior on the synchronization state.

1. Introduction

Ultrasound contrast agents (UCAs) are coated microbubbles by a stabilizing shell (polymer, albumin or lipid) which have the medical applications such as diagnostic ultrasound imaging and drug and gene delivery [1,2]. So far, most of the investigations have been devoted to the dynamics of the single microbubble. When UCAs interact with another one, the dynamical behavior of the interaction is completely different from the isolated case. Therefore, a good mathematical modeling of multi-microbubble dynamics in a cluster becomes extremely necessary. The study of radial dynamics of spherical single-bubble was introduced primarily by Rayleigh [3] which is formulated as free gas bubble in the incompressible inviscid liquid. Further studies by Plesset and Prosperetti [4,5] considered acoustical field for Rayleigh basic equation which called the Rayleigh-Plesset (R-P) model. A complete Rayleigh model is Rayleigh, Plesset, Noltingk, Neppiras, and Poritsky (RPNNP) equation [6,7] which include the effects of liquid viscosity, surface tension, and an incident acoustic pressure wave with low acoustic amplitude parameters. Later, Keller-Miksis [8] derived a model for free gas bubble in which the liquid's compressibility can be easily incorporated. The first UCAs model which added a thin viscoelastic albumin-shell and damping coefficient term to the RPNNP equation is proposed by de jong et al. [9]. The shell thickness and rigidity of UCAs in the RPNNP model were also considered by Church [10]. Multi-bubbles were theoretically studied by Takahira by means of the series expansion of the spherical harmonic (Legendre series) [11]. Doinikov by using the lagrangian formalism and Clebsch-Gordan expansion investigated a mathematical model for collective free gas bubble dynamics in strong ultrasound fields [12]. The cluster of microbubbles with a thin encapsulation added to the Keller-Herring (K-H) equation [13] have been analyzed by Macdonald and Gomatam [14]. The shape mode oscillations of microbubbles at high driving pressure have been reported in [15]. Moreover, it has been demonstrated in [16,17] that the effects of coupling between the bubbles can be significant when inter-bubble distances in a cluster are small.The nonlinear nature of above theoretical models need specialized tools for analysis due to the fact that linear and analytical solutions are not enough. When the motion of bubbles or UCAs gets chaotic, their theoretically observed behaviors with chaos theory tools such as bifurcation and Lyapunov diagrams [18–21] have been studied. For this reason, it is substantial to have appropriate information about the microbubbles dynamics, for finding an acceptable stability region in various applications in industry. Since the K-H model for UCAs is usually not studied in terms of *N* interacting microbubbles, the question arises, what distribution do

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the microbubbles correspond to?.

The concept of coupled map lattices (CML) was first suggested by Kaneko [22,23], which can be demonstrated as an array of smaller finite-dimensional subsystems endowed with local interactions. The CML consists of an array of dynamical elements which interacts (coupled) with other elements whose values are continuous or discrete in space and time [24,25]. Behaviors discovered in CML have been observed in chemical systems, fluids, electronics, traffic [26–28], optical systems, networks [29], and as well as in neural dynamics [30] and biological, and also in indirect experiments [31]. An extension of CML, in which each element is connected with all other elements is called globally coupled map (GCM) [32–34]. GCM have diverse applications in a real physical world such as Josephson junction arrays [35] and multi-mode lasers [36].

Significant strides have been made for studying the dynamics of network structures [37,38]. Gross et al. [39] specifically focused on the global synchronization among all oscillators. The authors of Ref. [40] analyzed the design of easily synchronized networks and found synchronizability to be varying. Synchronization is the most typical collective behavior in complex networks showing trajectories of each coupled dynamical elements which remains in step with each other during the temporal evolution. Complete synchronization is introduced by Pecora and Caroll in 1990 [41], where by means of synchronization, the state variables of individual systems converge towards each other [42]. One of the powerful mathematical technique which has been used by several authors [43–45] is the analysis of synchronization corresponding to the associative relation between the array of coupled oscillators and graph theory. Bose and Nair [46] in the design of statistical experiments introduced the theory of association schemes. In fact, association schemes as algebraic combinatorics are relations between pairs of elements of a set, which also arise naturally in the theory of permutation groups, independent of any statistical applications [46,47]. The governing algebra on the association schemes was formulated by Bose and Mesner [48] which is known as the Bose-Mesner algebra. Bose-Mesner algebra of an association scheme is the matrix algebra which generate then by adjacency matrices of the elements of the set. One is lead to ask two questions. The CML gives information of the stability region of which physical quantity? Can one investigate the dynamic behaviors of N interacting microbubble cluster from the CML approach at complete synchronization state?.

The answer to the first question depends on the physical context in which the CML is defined. Dynamics of coupled chaotic oscillators on the physical context are studied using coupled maps. The study of CMLs is one significant method to investigate the emergent phenomena, such as synchronization, cooperation, and more, which may happen in interacting physical systems. As a physical example, the chaotic nature of the microbubble-microbubble interaction requires particular tools for resolution, because the analytical and linear solutions are not sufficient.

Since the K-H model for UCAs is usually not studied in terms of N interacting microbubbles [14,21]. In this paper, to answer the above question, we employ an association scheme and the Bose-Mesner algebra in order to calculate the stability of N-microbubbles in the cluster at complete synchronization state. The coupling strength of the coupled K-H model could generate Markovian matrices and satisfy Markov conditions. In particular, when a cluster of microbubbles is globally synchronized, their dynamics are reduced that of a single microbubble. In this case, CML should be relevant for studying the salient behaviors of microbubble-microbubble interaction. In the present paper, we study complete synchronization of ultrasound contrast agents (UCAs) microbubbles interaction in a cluster. An important advantage of this method is that it is enough to have information only for one microbubble. All the numerical results demonstrate that association schemes perspective for studying the radial response of UCA microbubbles is very effective.

2. Definitions: Graph theory, association schemes and Bose-Mesner algebra

In this section, we give some preliminaries such as definitions related to graph theory, association schemes and Bose-Mesner algebra which are used through out the paper [49,50].

Graph is a pair $\Omega = (V,E)$, where V is a non-empty set(Ω) and E is a subset of $\{(\alpha,\beta)\colon \alpha,\beta \in V, \alpha \neq \beta\}$. Elements of the graph are called vertices (V) and edges (E). Two vertices $\alpha,\beta \in V$ are called adjacent if $\{\alpha,\beta\} \in E$. The adjacency matrix is defined by [51,52],

$$(A)_{i,j} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Obviously, A is symmetric matrix. The valency of a vertex, $i \in V(G)$ is defined as

$$deg(i) \equiv \kappa(i) = |\{j \in V(G)\colon i \sim j\}|$$

where $|.|$ denotes the cardinality (the cardinality of a set is a measure of the number of elements of the set). Let V be a set of vertices and $R_\alpha = \{R_0, R_1, \ldots, R_d\}$ be a nonempty set of relations on V which is named associate class. The pair $\widetilde{X} = (V, R_\alpha)$ is called an association scheme of class d (d-class scheme) on V under the following four conditions [53,49,50]:

1. $\{R_\alpha\}$ is a part of $V \times V$,
2. $R_0 = \{(i,i)\colon i \in V\}$,
3. $R_\alpha = R_\alpha^T$ where $R_\alpha^T = \{(j,i)\colon (i,j) \in R_\alpha\}$,
4. For any $(i,j) \in \beta$, the number of $p_{\alpha\beta}^{\gamma} = |\{k \in V\colon (i,k) \in R_\alpha \text{and } (k,j) \in R_\beta\}|$ depends only on α,β,γ.

where $\widetilde{X}$ is a symmetric and commutative association scheme of class d from conditions (3) and (4), respectively. The elements i and j of V are called α^{th} associates if $(i,j) \in R_\alpha$ and $d+1$ is the number of associate classes which is called the rank of the scheme. The intersection numbers of the association scheme are denoted by $p_{\alpha\beta}^{\gamma}$. Indeed, condition (3) implies that $p_{\alpha\beta}^{0} = 0$ if $\alpha \neq \beta$ while $p_{0\beta}^{\beta} = p_{\alpha 0}^{\alpha} = 1$. Also, condition (4) implies that every element of V has $p_{\alpha\alpha}^{0}$ which is defined as

$$\kappa_\alpha = p_{\alpha\alpha}^{0} \tag{1}$$

this is called the valency of α^{th} associates class ($\kappa_\alpha \neq 0$). Relation between the number of vertices (or order of the association scheme) and valency is given by:

$$N = |V| = \sum_{\alpha=0}^{d} \kappa_\alpha \tag{2}$$

other definition are given as;

$$\sum_{\alpha=0}^{d} A_\alpha = J_N, \quad A_0 = I_N,$$
$$A_\alpha = A_\alpha^T, \quad A_\alpha A_\beta = \sum_{\gamma=0}^{d} p_{\alpha\beta}^{\gamma} A_\beta. \tag{3}$$

J is an $N \times N$ matrix with all-one entries. Also, a sequence of matrices $A_0, A_1, \ldots, A_d$ generates a commutative $(d+1)$−dimensional algebra $\mathbf{A}$ of symmetric matrices which is called Bose-Mesner algebra of $\widetilde{X}$ [48]. It should be noted that, the matrices A_α are commuting and they can be diagonalized simultaneously [54]. There exists a matrix (M) in such a way that for each, $A \in \mathbf{A}, M^{-1}AM$ is a diagonal matrix. Therefore, $\mathbf{A}$ has a second basis $E_0, \ldots, E_d$ [51,55]. These matrices satisfy

$$E_0 = \frac{1}{N} J_N, \quad E_\alpha E_\beta = \delta_{\alpha\beta} E_\alpha, \quad \sum_{\alpha=0}^{d} E_\alpha = I_N \tag{4}$$