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# T-stress solution of penny-shaped cracks in transversely isotropic elastic media

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ABSTRACT

This paper presents explicit formulae of the T-stress for a penny-shaped crack in a transversely isotropic, linearly elastic, infinite medium under remote triaxial stress and crack-face normal traction. The T-stress Green's function for a pair of opposite unit point forces is established first in a closed-form and these results are then utilized to derive the integral formulae of the T-stress for a penny-shaped crack under general loading conditions. The explicit dependence of the T-stress on material properties for both isotropic and transversely isotropic solids is presented. The closed-form expressions of the T-stress for uniform remote triaxial stresses and uniform, linear, and axisymmetric normal tractions are also reported.



## 1. Introduction

Existing, well-established mathematical models based upon the classical theories of linear elasticity and linear elastic fracture mechanics have been found to be sufficient, well-suited, and commonly employed to perform stress analysis of bodies containing defects/flaws to provide essential information in damage/fatigue assessments. Modeling of cracks and their advances is often based on a dominant stress field in a local region surrounding the crack front—a singular term in the series representation of the near-front stress field (e.g., [23]). It has been well recognized that such a singular term can be completely characterized by a single set of parameters termed the stress intensity factors along with the known angular variation resulting from the eigen analysis. One obvious application of the dominant stress field, in addition to the prediction of crack initiation and direction of crack advances (e.g., [18,3,10]), is the estimation of size and shape of the plastic zone surrounding the crack front (e.g., [1]). This information plays an important role in the classification of fracture problems into small-scale or large-scale yielding. However, it has been noted by various investigators that merely the stress intensity factors are inadequate for the accurate prediction of the plastic zone size and shape; it additionally requires the first nonsingular term in the representation of the near-front stress field—the T-stress (e.g., [15,2]). The crucial role of the T-stress was also evident in the investigation of fracture initiation angle, the triaxiality of the near-front stress field, and the stability of the crack propagation as indicated by Sedighiani et al. [17]. In particular, the presence of the positive T-stress along the crack front generally reduces the plastic zone size and rotates its shape backward while increasing the initiation angle of crack advances and strengthening the crack-tip triaxiality compared with those predicted by the singular stress term only; in contrast, the negative T-stress reverses all of those effects. From past evidences, the integration of the T-stress information in fracture modeling becomes essential and, as a direct consequence, an accurate calculation of these fracture data along the crack front is

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**Nomenclature**

| | |
|---|---|
| $a$ | radius of penny-shaped crack |
| $A_{11}, A_{13}, A_{33}, A_{44}, A_{66}$ | elastic constants of transversely isotropic materials |
| $\bar{A}_{13}, \bar{A}_{33}, \bar{A}_{44}, \bar{A}_{66}$ | normalized elastic constants of transversely isotropic materials |
| $i = \sqrt{-1}$ | imaginary number |
| $\mathbf{I}(f)$ | imaginary part of a complex function $f$ |
| $l, \phi$ | transformed variables related to $r_0, \theta_0, \theta$ and radius of the crack |
| $m_1, m_2, H$ | material-dependent parameters |
| $\{\boldsymbol{O}; x_1, x_2, x_3\}$ | global reference Cartesian coordinate system |
| $\{\boldsymbol{O}; r, \theta, z\}$ | global cylindrical coordinate system |
| $P$ | resultant of applied axisymmetric normal traction $t(r_0, \theta_0) = f(r_0)$ |
| $P_a$ | resultant of uniform normal traction with the magnitude $f(a)$ |
| $\bar{r}, \bar{\theta}$ | local polar coordinates on $\bar{x}_1 - \bar{x}_2$ plane |
| $(r_0, \theta_0, 0^{\pm})$ | global cylindrical coordinates of point on the upper and lower crack surfaces where normal concentrated forces are applied |
| $\mathbf{R}(f)$ | real part of a complex function $f$ |
| $t^0$ | applied normal traction on the crack surface |
| $T_{ij}(\boldsymbol{x}_c)$ | components of T-stress tensor at a point $\boldsymbol{x}_c$ |
| $T_{11}, T_{13}, T_{33}$ | unknown components of T-stress tensor |
| $T_{12}, T_{22}, T_{23}$ | known components of T-stress tensor |
| $T_{\Lambda}$ | T-stress components |
| $T^G_{\Lambda}$ | T-stress Green's function |
| $T^{\infty}_{\Lambda}$ | T-stress components due to remote triaxial stress |
| $T^0_{\Lambda}$ | T-stress components due to uniform part of normal traction |
| $T^1_{\Lambda}$ | T-stress components due to non-uniform part of normal traction |
| $u_i$ | displacement components referring to $\{\boldsymbol{O}; x_1, x_2, x_3\}$ |
| $\boldsymbol{x}_c$ | point along the crack front |
| $\{\boldsymbol{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ | local reference Cartesian coordinate system |
| $\varepsilon_{ij}$ | strain components referring to $\{\boldsymbol{O}; x_1, x_2, x_3\}$ |
| $\gamma_1, \gamma_2, \gamma_3, \Upsilon, \Gamma, \Phi$ | material-dependent parameters |
| $v$ | Poisson's ratio |
| $\boldsymbol{\sigma}^{\infty}$ | remote triaxial stress |
| $\boldsymbol{\sigma}^G$ | stress Green's function associated with a pair of self-equilibrated, unit normal concentrated forces acting to the surface of a penny-shaped crack |
| $\sigma_{ij}, \bar{\sigma}_{ij}$ | stress components referring to $\{\boldsymbol{O}; x_1, x_2, x_3\}$ and $\{\boldsymbol{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$, respectively |
| $\sigma^{\infty}_{ij}, \bar{\sigma}^{\infty}_{ij}$ | components of $\boldsymbol{\sigma}^{\infty}$ referring to $\{\boldsymbol{O}; x_1, x_2, x_3\}$ and $\{\boldsymbol{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$, respectively |
| $\bar{\sigma}^K_{ij}, \bar{\sigma}^T_{ij}, \bar{\sigma}^m_{ij}$ | radial independent functions in near-front expansion of stress field referring to $\{\boldsymbol{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ |
| $\bar{\sigma}^G_{ij}$ | components of $\boldsymbol{\sigma}^G$ referring to $\{\boldsymbol{x}_c; \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ |
| $\sigma^G_{rr}, \sigma^G_{\theta\theta}, \sigma^G_{zz}, \sigma^G_{r\theta}, \sigma^G_{rz}, \sigma^G_{\theta z}$ | components of $\boldsymbol{\sigma}^G$ referring to $\{\boldsymbol{O}; r, \theta, z\}$ |

required. Studies toward the development of solution techniques for calculating the T-stress and the investigation of its influence on the fracture responses and behavior have been carried out by many researchers in the past several decades. Some relevant literature is briefly summarized below to shed some light on the historical background and current advances in the area and finally reflect the novel aspect of the present study.

In 1974, Rice originally investigated the influence of the T-stress on the estimation of the plastic zone size and shape using the Barrenblatt–Dugdale yielding model and confirmed that the T-stress significantly affects both the size and shape of the plastic zone surrounding the crack tip. Later, Du and Hancock [2] applied the von Mises yielding criterion to explore the role of the T-stress in the calculation of the plastic zone size and shape for a plane strain crack, and they also concluded that the plastic zone is enlarged and rotates forward for the negative T-stress and is shrunk and rotates backward for the positive T-stress. Fett [5] calculated the T-stress of an edge-cracked, rectangular, finite plate made from an isotropic, linearly elastic material by using boundary collocation and Green's function techniques. Fundamental results for a pair of normal concentrated forces were developed first and then applied to generate solutions for prescribed arbitrary normal tractions. Later, Fett [6] extended his earlier work to obtain numerical results for the T-stress of an edge crack and a center crack in isotropic, linearly elastic, rectangular plates and circular disks subjected to both tensile and bending loads. Wang [19] adopted the weight-function technique and the finite element method to determine the T-stress for various cases including a single edge-cracked plate (SECP), a double edge-cracked plate (DECP) and a center-cracked plate (CCP) under uniform, linear,