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On three-dimensional asymptotic solution, and applicability of Saint–Venant's principle to pie-shaped wedge and end face (of a semi-infinite plate) boundary value problems

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Dedicated to Dr. Nicholas J. Pagano, Former Senior Scientist (ST), AFRL, Wright-Patterson AFB, Ohio, on the occasion of his 80th birthday.

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ABSTRACT

Three-dimensional asymptotic singular stress fields near the fronts of infinite wedges are presented. This investigation is devoted to establishment of rigorous conditions as to whether the presence of wedge or end face stress singularity, which depends on the prescribed boundary condition, can validate or invalidate the heuristic assumption implicit in Saint–Venant's principle. This is accomplished by applying a general approach to the solution of canonically singular problems, based on the concept of proper boundary-value problem, the theorem of homogeneous solutions, and classification of boundary value problems (BVP) of three-dimensional elasticity theory into class S (Saint–Venant) or class N (non-Saint–Venant).

1. Introduction and Background

Since Williams' [1] study pertaining to the singular stress field at a wedge tip, wedge problems, with homogeneous boundary conditions, have been extensively studied under the premises of two-dimensionality (see e.g., Ref. [2]). Gregory [3] has proved the completeness of Papkovich-Fadle eigenfunctions for a rectangle and Williams eigenfunctions for an annular wedge shaped sector via corresponding bilinear representations of the Green's functions for these regions for fields with finite energy or bounded displacements. The mathematical difficulties posed by the three-dimensional wedge problems (with homogeneous boundary conditions) are substantially greater than their two-dimensional counterparts. One of the objectives of the present investigation is to address this important issue in the context of singular boundary value problems.

Singular boundary value problems (i.e., boundary value problems with singular points or lines) of the two- or three-dimensional theory of linear elasticity include cracks and re-entrant wedges, concentrated force, dislocations, disclinations, etc. Singular points for two-dimensional elasticity include, e.g., an infinitely remote point, a corner point, a conical

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Nomenclature

E, ν, G	Young's modulus, Poisson's ratio and shear modulus, respectively
$\hat{\lambda}$, G	Lame's coefficients
p	exponent of the θ-dependent parts of the variable-separated assumed displacement functions, U_r, U_θ, U_z
r, θ, z	cylindrical polar coordinate system
$R_r(r), R_\theta(r), R_z(r)$	r-dependent parts of the variable-separated assumed displacement functions, U_r, U_θ, U_z
U_r, U_θ, U_z	components of the displacement vector in polar coordinates
α	constant, called the wave number
λ	order of stress singularity
θ	half the wedge aperture angle

point, a point of discontinuity of the boundary conditions, a point of application of a concentrated force, etc. Since the boundary value is not defined at a singular point (line for the three-dimensional case), the formulation of a physically meaningful additional condition at this point (line for the three-dimensional case) then becomes a necessity, which calls for a proper singular boundary-value problem of two- and three-dimensional linear elasticity theory in a manner first expounded by Cherepanov [4].

In regards to an infinitely remote line or point, Saint–Venant [5] was the first to introduce a heuristic "principle of the elastic equivalence of statically equipollent systems of loads" for solving the problems of torsion and bending of elastic rods or cylinders [6]. Saint–Venant's approach is based on the conjecture that his approximate solution would be valid far enough away from the ends of a sufficiently long cylinder with traction-free lateral surfaces, in situations where the end tractions are statically equivalent to, but not identical with those for which his solution is rigorously valid. Love [7] states, "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with linear dimensions of the part." For a more precise definition and a rigorous treatment of the Saint–Venant principle, see Sternberg [8] and Knowles [9].

The fact that the problem relating to Saint–Venant's principle has attracted so much attention for over a century and half (see e.g., [10–12] for the bibliography) testifies to its importance from both theoretical and practical points of view. From the practical standpoint, such a principle is intuitively invoked in almost routine manners in engineering applications, especially in devising a simple tension test for a material, by clamping the ends of the test specimen in the jaws of a testing machine with the implicit assumption that the action on the central part of the bar (i. e., the gauge section) is nearly the same as if the forces were uniformly applied at the ends [6]. Validity of this assumption is justified when the specimen fails in the gauge section, which is generally the case with carefully prepared test specimens with end tabs. But unfortunately, end failures are not uncommon in a laboratory testing environment, which invalidates the universal applicability of such an assumption. This phenomenon is generally considered to result from the presence and interactions of geometric discontinuities, such as crack-like flaws or notches, and occurs within a very localized region near the geometric boundaries of a plate. Additional complexity arises from a line of discontinuity of the boundary conditions, and its interaction with other geometric discontinuities. For example, the effects of stress singularity in the neighborhood of the circumferential re-entrant corner lines of the elliptical/circular cylindrical internal and surface flaws, weakening laminated composite plates, have been numerically assessed by Chaudhuri [13,14] and Chaudhuri et al. [15]. The interaction of this singularity with end or edge face (free edge in two dimension) stress singularity at the plate boundary, and the implication of such interactions (i.e., violation of Saint–Venant's principle) in regards to testing of laminated composite specimens have also been thoroughly investigated [13–15]. For homogeneous plates weakened by similar internal and surface part-through flaws, there is no violation of Saint–Venant's principle, because of absence of such an edge face singularity [14,16,17]. This notwithstanding, a rigorous theoretical basis of such violations or lack thereof is still absent in the literature, which is an important focus of the present study.

On the theoretical side, the seminal paper by Sternberg and Koiter [18] and follow-up studies such as Dundurs and Markenscoff [19], and Markenscoff [20] called into question the assumed universal validity of the intuitive order of affairs envisaged by the Saint–Venant principle in the context of the two-dimensional "wedge paradox" problem. The problem of an infinite two-dimensional wedge loaded by a concentrated couple at its vertex was first solved by Carothers [21]. The Carothers solution, which corresponds to the displacement function eigenvalue of -1 (independent of the wedge aperture angle), breaks down at a critical angle. Furthermore, as Markenscoff [20] has pointed out, at the critical angle the Carothers stress field is self-equilibrated not only for the force but also for the resultant couple on r = constant surfaces. The Saint–Venant principle does not hold for the critical angle, because of the requirement that the stresses decay faster than r^{-2} as $r \to \infty$, when the applied distributed loading has vanishing stress and moment resultants [20].

Cherepanov [4] has presented a general approach to the solution of singular points, based on the concept of proper boundary-value problem and the theorem of homogeneous solutions. For the plane (two-dimensional) problems of the theory of elasticity, the entire set of singular elastic problems with infinitely remote points are classified by Cherepanov [4] into (i) Class S wherein Saint–Venant's principle applies, and (ii) Class N, which does not obey Saint–Venant's principle. In the latter case, in order to state a correct boundary-value problem, it is necessary to introduce an additional condition at infinity.