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Three-dimensional elastic stress fields ahead of notches in thick plates under various loading conditions

Michele Zappalorto, Paolo Lazzarin *

University of Padova, Department of Management and Engineering, Stradella S. Nicola 3, 36100 Vicenza, Italy

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ABSTRACT

The paper discusses the features of three-dimensional elastic stress distributions ahead of notches in finite thick plates under different loading conditions. It is proved that, under certain circumstances, the three-dimensional governing equations of elasticity can be reduced to a system where a bi-harmonic equation and a harmonic equation have to be simultaneously satisfied. The former provides the solution of the corresponding plane notch problem, the latter provides the solution of the corresponding out-of-plane shear notch problem. The analytical frame is applied to a number of notched geometries, and its degree of accuracy is discussed comparing theoretical results and numerical data from 3D FE models. Practical consequences on the expected crack paths are also documented for one of the considered models.

1. Introduction

The importance of analytical solutions for stresses in solids has been recognised for many years. However, the inherently complicated nature of three-dimensional stress distributions near notches or cracks has urged many authors to devote their efforts to plane cases (plane stress or plane strain), whilst there are comparatively few analytical solutions available for three-dimensional stress concentration problems.

Lamè [1] studied the stress fields around a spherical cavity under internal or external uniform pressure, the extension to the tension loading case being due, instead, to Southwell and Gough [2]. Goodier's solution [3] for the stress fields around an elastic spherical inclusion under tension remains a milestone; it has also been reconsidered and extended to the calculation of the adhesion strength at the interface of coated particles and to nanoparticles [4,5]. Worth of mentioning are also the solutions by Sternberg and Sadowsky [6] and Eshelby [7] dealing with the triaxial ellipsoidal cavity and the ellipsoidal inclusion, respectively.

An approach providing a general solution for three-dimensional problems of elasticity, based on a general three-dimensional stress function composed by three harmonic functions, was developed by Papkovich [8] and, independently, by Neuber [9]. Neuber [9] extensively used this method in combinations with appropriate curvilinear coordinate systems and provided a number of solutions for two-dimensional notch problems and some solutions for the three-dimensional problems, such as the axisymmetric hyperboloidal ligament and the axisymmetric ellipsoidal cavity in an infinite elastic body. The types of loading were pure tension, pure bending, pure shear and pure torsion. It should be pointed out that the geometry is axisymmetric, although the bending and shear are not symmetric. Neuber also discussed some cases of spatial notch geometries where the state of stress could be conveniently reconverted to a simple two-dimensional case [9]. Recent papers

* Corresponding author.
E-mail address: plazzarin@gest.unipd.it (P. Lazzarin).

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Nomenclature

2W	plate width
E	modulus of elasticity in tension
G	modulus of elasticity in shear
K_1, K_2, K_3	notch stress intensity factors, NSIFs, under Modes I, II and III
t	plate thickness
u_x, u_y, u_z	displacement components in the Cartesian coordinate system
x, y, z	Cartesian coordinates
2α	notch opening angle
ε_{xx}, ε_{yy}, ε_{zz}	strains in the x, y and z directions
ϕ	Airy's stress function
γ	$\pi-\alpha$
γ_{xy}, γ_{zx}, γ_{zy}	shear strain components in rectangular coordinates
λ_1, λ_2	Mode I and Mode II Williams' eigenvalues
$\lambda_{3,s}$, $\lambda_{3,a}$	symmetric and skew-symmetric eigenvalues for Mode III (antiplane mode)
v	Poisson's ratio
r, θ, z	polar coordinates
ρ	notch root radius
σ_{xx}, σ_{yy}, σ_{zz}	normal stress components in rectangular coordinates
σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}	normal stress components in polar coordinates
$\tilde{\sigma}_{xx}, \tilde{\sigma}_{yy}, \tilde{\sigma}_{zz}$	in-plane variation of the normal stress components in rectangular coordinates
$\sigma_{\theta\theta,\mathrm{tip}}$	hoop stress, $\sigma_{\theta\theta}$, evaluated at the notch tip
τ_n	nominal shear stress
$\tau_{zy,\mathrm{tip}}$	shear stress, τ_{zy}, evaluated at the notch tip
τ_{xy}, τ_{zx}, τ_{zy}	shear stress components in rectangular coordinates
$\tau_{r\theta}$, τ_{zr}, $\tau_{z\theta}$	shear stress components in polar coordinates
$\tilde{\tau}_{xy}, \tilde{\tau}_{zx}, \tilde{\tau}_{zy}$	in-plane variation of the shear stress components in rectangular coordinates

have shown that a comparatively simple state of stress is present also near circumferential notches in axi-symmetric bars under torsion (see [10,11] and references reported therein).

An analytical frame including the bases of the Papkovich–Neuber's method was proposed *ante litteram* by Dougall [12] to evaluate the three-dimensional stress fields in plates. Dougall's pioneering method was later applied by Green [13] to the stress analysis of a thick isotropic elastic plate weakened by a cylindrical hole. The same problem was discussed later also by Sternberg and Sadowsky [14] and by Folias and Wang [15]. The common denominator of these works is the finding that the plate thickness exerts a non-negligible influence on the hole tip stress.

When the stress concentration in the plate is caused by configurations with stress singularities, like those due to re-entrant corners or sharp cracks, the problem is considerably more difficult to be analysed. Dealing with the crack case, Hartranft and Sih [16] gave the approximate stress fields near the tip of a through crack in a thin elastic plate using a variational principle. Other researchers used various simplifying assumptions to reduce the 3D analysis to manageable proportions. For example, Williams dealt with the stress analysis of the out-of-plane bending of V-notched plates [17] using the Kirchhoff thin plate theory, which was later applied also to cracked plates [18]. In the last case, Williams found that the elastic near-tip stresses continued to present the inverse square root singularity. Due to the approximate nature of the fourth-order Kirchhoff theory, Williams' solution [18] had some drawbacks relating to the out-of-plane shear stresses. As is noted in the review by Zehnder and Viz [19], these drawbacks were not present in the solution provided by Knowles and Wang [20] and based on the Reissner sixth-order theory which takes into account the effect of the transverse shear strains. A further refinement of the Reissner theory is due to Hartranft and Sih [21], who were also able to extend in Ref. [22] Williams' two dimensional eigenfunction series [23] to the three-dimensional crack case. Some years later, Kassir and Sih [24] used Papkovich–Neuber's method to address the same problem. The results from William's analysis on angular corners in bending plates [17] are the starting point of a recent contribution by Huang [25], who has investigated the singular behaviour of moments, shear forces and in-plane forces, making use of the Lo's high-order plate theory.

Instead of bending plate theories, the Kane and Mindlin hypothesis [26] was used by Yang and Freund [27] to analyse the state of stress in a tensioned thin elastic plate containing through-cracks. By using the Fourier transform and the Wiener–Hopf technique, they demonstrated the existence of a generalised plane strain field at the crack tip, and confirmed Hartranft and Sih's previous findings [16] according to which the angular variations of "in-plane stresses" σ_{xx}, σ_{yy}, and τ_{xy} equate those of the plane elastic crack problem. The same results were also numerically found by Nakamura and Parks [28]. The Kane and Mindlin theory was used by Kotousov and Lew [29] to analyse the stress singularities related to angular corners in plates of arbitrary thickness with various boundary conditions subjected to in-plane loadings. The corresponding singular stress states were referred to as the out-of-plane singularities and the corresponding fracture mode as the out-of-plane singular mode or K_O-mode.