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Computing stress singularities in transversely isotropic multimaterial corners by means of explicit expressions of the orthonormalized Stroh-eigenvectors

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ABSTRACT

Composite materials reinforced by unidirectional long fibers behave macroscopically as homogeneous transversely isotropic linear elastic materials. A general, accurate and computationally efficient procedure for the evaluation of singularity exponents and singular functions characterizing singular stress fields in multimaterial corners involving this kind of material is presented in this paper. To take full advantage of the sextic Stroh formalism of anisotropic elasticity applied to this particular problem, the complete set of explicit expressions of the eigenvalues and eigenvectors of the real 6×6 fundamental elasticity matrix **N** has been deduced for all the non-degenerate and degenerate (repeated roots of the sextic Stroh equation) cases. These expressions will also facilitate further applications of the Stroh formalism to these materials. Several numerical examples of singularity analysis of multimaterial corners appearing in adhesively bonded joints and damaged cross-ply laminates of composite materials are presented.

1. Introduction

The linear theory of elasticity predicts, in general, singular (unbounded) stresses at discontinuities in geometry, material properties and boundary conditions. Configurations where different materials converge at one point can be easily found, for instance, in composite structures, adhesively bonded joints and microelectronics. These configurations, called *multimaterial corners* or cross points, are potential places where failure can initiate due to these singular stress fields.

In the present paper special interest is focused on piecewise homogeneous multimaterial corners involving transversely isotropic materials, representing composite laminas reinforced by unidirectional long fibers, and subjected to a generalized plane strain state. These laminas play an important role in composite applications due to their high specific stiffness and strength.

According to a general mathematical formulation of a corner problem, which can be found in Kondratev [1], Costabel and Dauge [2] and Nicaise and Sändig [3], the displacement components u_i ($i = 1,2,3$) in the neighbourhood of the corner tip, where a polar coordinate systems (r,θ) is considered, can be written as a sum of terms, each one of them having the following form:

$$u_i(r,\theta) = Kr^{\delta}\log^{p} r g_i(\theta,\delta,p), \quad p = 0,1,\ldots \tag{1}$$

where (real or complex) numbers δ are called *singularity exponents* and functions g_i are called *singular* (or *characteristic*) *functions*. Both δ and g_i depend only on the local corner configuration (geometry, materials and boundary conditions at the corner

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tip) and are usually evaluated by means of a quadratic or a nonlinear eigenvalue problem, where δ are given by eigenvalues and g_i are obtained from the associate eigenvectors. The logarithmic terms are present in (1) if and only if the algebraic multiplicity of an eigenvalue is greater that its geometric multiplicity. This happens, for instance, in the transition from two real to two complex conjugates roots, a very particular case that is not considered in the present study, only non logarithmic singularities being considered in what follows.

In the asymptotic series expansion of an elastic solution in the corner tip neighbourhood, the coefficients K of the terms in the form (1) for displacements, and of the corresponding terms for stresses, are called *generalized stress intensity factors*, which depend on the overall problem formulation (the global domain, materials and boundary conditions).

A general, accurate and computationally efficient procedure for the evaluation of singularity exponents δ and singular functions g_i should: (a) cover all kind of homogeneous transversely isotropic linearly elastic materials at any spatial orientation, (b) consider any finite number of homogeneous wedges converging at the corner tip and perfectly bonded between them, (c) consider all kind of standard homogeneous boundary conditions, and (d) provide, in general, expressions as most analytic and compact as possible.

Although in the past several general excellent procedures for singularity analysis of anisotropic linear elastic multimaterial corners have been developed, none of them combined all the above listed features. Let us mention, at least, a few outstanding computational procedures implemented: Leguillon and Sanchez-Palencia [4] constructed eigenvalue problems by a finite element discretization in the angular variable; Papadakis and Babuška [5] obtained eigenvalue problems by using a numerical solver for systems of ordinary differential equations and a shooting technique; Yosibash [6] applied finite element discretizations of a modified Steklov formulation in an angular sector of an annulus; Costabel et al. [7] particularized the general solution basis for the corresponding system of ordinary differential equations in the angular variable deduced in [2] to the elastic corner problem, and constructed the eigenvalue problems in an analytic way, with an exception of the evaluation of the roots of a characteristic equation for each homogeneous material included in the corner. While the former three procedures are numerically based, not working with a closed form expression of the eigenvalue problem, the later one is analytically based, starting from a closed form representation of the elastic solution at a corner and presenting a closed form expression of the eigenvalues problem. Any analytically based procedure for the corner singularity analysis is expected to be faster than numerically based ones and allowing also a more detailed analysis of the position and nature of singularity exponents (eigenvalues), in particular the relation of its algebraic and geometric multiplicities. A disadvantage of the procedure developed by Costabel et al. [7] is that it does not cover all kind of anisotropic materials (in particular not the degenerate ones) and that the size of the eigenvalue problem depends on the number of homogeneous wedges included in the corner.

The present paper is aimed to develop a general procedure which satisfies the above mentioned requirements for the evaluation of singularity exponents δ and singular functions g_i.

It appears that the most suitable analytic approach to formulate the eigenvalue problem corresponding to a multimaterial corner in a compact form is based on the so-called Lekhnitskii–Eshelby–Stroh complex variable formalism of two dimensional anisotropic elasticity (see Lekhnitskii [8], Eshelby et al. [9], Stroh [10,11], (see Ting [12], for a comprehensive review), and in the following referred to as Stroh formalism. Ting [13], Wu [14], Barroso et al. [15], Yin [16] and Hwu et al. [17] showed that this formalism is a powerful analytical tool for the singularity analysis including anisotropic materials, and thus it also provides the theoretical basis for the present work.

The elastic solution representation in the Stroh formalism is based on knowledge of the eigenvectors of the real 6×6 *fundamental elasticity matrix* **N**, which is associated with a specific orientation of the coordinate system with respect to the material. To take full advantage of the fundamental orthogonality and closure relations of the Stroh formalism the eigenvectors have to be properly orthogonalized and normalized.

Several authors have studied the Stroh formalism applied to transversely isotropic materials. Tanuma [18] obtained the surface impedance tensor (associated with the linear relationship between displacements and tractions given at a surface), and Nakamura and Tanuma [19] and Ting and Lee [20] introduced independently new explicit closed forms of the 3D fundamental solution; for further developments of this solution see Távara et al. [21]. The complete set of explicit expressions of the orthonormalized eigenvectors is, to the authors' knowledge, not available for these materials at present. This is perhaps because, although it is possible to deduce them in a relatively straightforward way, the deduction is somewhat tedious as requires lengthy algebraic calculations. Tanuma [18] obtained explicit expressions of some of the Stroh-eigenvectors for transversely isotropic materials in the dual coordinate systems as intermediate results in his procedure for the evaluation of the surface impedance tensor. Nevertheless, not all of them (the eigenvectors) were obtained in Tanuma's work (it was not necessary for the final result) and those obtained were neither normalized nor orthogonalized (again because it was not necessary for the final result). It is also important to mention that many applications do need the orthonormalized expressions of the Stroh-eigenvectors in the coordinate system used in the present work and although the relationship between both coordinate systems is straightforward, lengthy and involved calculations are required for the final expressions to be explicitly obtained, as mentioned above.

Tanuma [18] showed how the relative orientation of a transversely isotropic material can make the matrix **N** mathematically degenerate (some of the eigenvalues are equal and the associate eigenvectors linearly dependent), leading to numerical instabilities when a critical relative orientation is approached, due to the fact that the eigenvectors are not continuously defined in the transition from the non-degenerate to degenerate cases. The explicit expressions of the eigenvectors obtained in the present work include, in addition to the mathematically non-degenerate cases, all mathematically degenerate cases