European Journal of Mechanics A/Solids 30 (2011) 854–864

Contents lists available at ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

Size effects on the plastic collapse limit load of thin foils in bending and thin wires in torsion☆

Castrenze Polizzotto

Università di Palermo, Dipartimento di Ingegneria Civile, Ambientale e Aerospaziale, Viale delle Scienze, 90128 Palermo, Italy

ARTICLE INFO

Article history:
Received 9 November 2010
Accepted 10 May 2011
Available online 20 May 2011

Keywords:
Gradient plasticity
Size effects
Plastic limit analysis

ABSTRACT

Following a previous paper by the author [Strain gradient plasticity, strengthening effects and plastic limit analysis, Int. J. Solids Struct. 47 (2010) 100–112], a nonconventional plastic limit analysis for a particular class of micron scale structures as, typically, thin foils in bending and thin wires in torsion, is here addressed. An idealized rigid-perfectly plastic material is considered, which is featured by a *strengthening potential* degree-one homogeneous function of the effective plastic strain and its spatial gradient. The nonlocal (gradient) nature of the material resides in the inherent *strengthening law*, whereby the yield strength is related to the effective plastic strain through a second order PDE with associated higher order boundary conditions. The peculiarity of the considered structures stems from their geometry and loading conditions, which dictate the shape of the collapse mechanism and make the higher order boundary conditions on the (microscopically) free boundary be accommodated by means of a boundary singularity mechanism. This consists in the formation of thin boundary layers with unbounded stresses, but bounded stress resultants which —together with the regular bulk stresses— contribute to the value of the collapse load. Closed-form solutions are provided for thin foils in pure bending and thin wires in pure torsion, and in particular the limit bending and torque moments are given as functions of an adimensionalized internal length parameter.

1. Introduction

Materials in the form of micron scale specimen, or under a highly localized strain field, are known to exhibit notable size effects, in the sense that "smaller is stronger". Widely recognized examples of such effects are: the increase of the indentation hardness with the decreasing of the indentation depth; the increase of the hardening rate with the decreasing of the specimen size; the so-called "strengthening effects", that is, the increase of the yield strength with the decreasing of the crystal grain size (Hall–Petch effects), or even of the specimen size. See Fleck and Hutchinson (1997), Hutchinson (2000), Aifantis (2003), Gudmundson (2004), Hansen (2004) and Gurtin and Anand (2005) for an overview on these phenomena and the related literature.

Another kind of strengthening effects, recently pointed out by the author (Polizzotto, 2010a) on theoretical bases, is the increase of the plastic collapse limit load of structures with the decreasing of the specimen size. In the latter quoted paper, plastic limit analysis for micron scale structures was addressed. It was shown that the basic concepts of the classical plastic limit analysis, including the concept of rigid-plastic behavior, can be extended to the mentioned structures in a way suitable to capture size effects (or strengthening effects). Like in classical limit analysis, for every loaded body, a plastic limit state exists, in which the body deforms plastically under constant load, the value of which depends on the *actual yield strength* (σ_y), the latter being determined through a *strengthening law* in terms of the effective plastic strain field. Since the strengthening effects, being strictly related to the shape of the collapse mechanism, are not known in advance, their evaluation needs to solve a nonconventional plastic limit analysis problem, in which the actual yield strength constitutes an additional unknown field carrying in the nonlocality features of the problem.

In the latter theory, the strengthening effects are simulated by means of a fictitious isotropic hardening featured by a *strengthening potential*, say $\psi_{st} = \psi_{st}(\kappa, \nabla\kappa)$, which is a *positively degree-one homogeneous* function of the effective plastic strain, κ, and its spatial gradient, $\nabla\kappa$. Consistent with nonlocal continuum thermodynamics, the above potential leads to the mentioned *strengthening law*, whereby a *strengthening stress*, say *Y*, expressing the increase of the initial yield strength at the generic point of the body is related to κ through a second order PDE (partial differential equation) with suitable higher order boundary conditions. The equation set

☆ This paper is dedicated to Prof. Giulio Maier on the occasion of his eightieth birthday.

E-mail address: c.polizzotto@unipa.it.

0997-7538/$ – see front matter © 2011 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.euromechsol.2011.05.003

governing the plastic limit analysis problem includes the above PDE and boundary conditions, beside all the equations pertaining to the classical counterpart problem (i.e. equilibrium and compatibility equations, yield conditions, normalization condition), but with the known initial yield strength, σ_{y0}, replaced by $\sigma_y = Y + \sigma_{y0}$.

The resulting nonclassical plastic limit analysis problem proves to be mathematically more complex than the classical one due to a coupling between the variables of static-type (load multiplier, stresses, strengthening stress) and those of kinematic-type (plastic strains, displacements) induced by the strengthening law, not present in the classical counterpart problem. Nevertheless, the fundamental theorems of classical plastic limit analysis, i.e. the solution uniqueness and the lower and upper bound theorems, were shown to hold in the presence of strengthening effects, such that the collapse load multiplier can still be obtained either as the maximum statically and plastically admissible load multiplier, or as the minimum kinematically admissible load multiplier (Polizzotto, 2010a).

Limit analysis problems like the one described above were addressed by Anand et al. (2005) for a one-dimensional shear model by means of a FEM (finite element method) procedure. The same shear model was addressed by Polizzotto (2010a) by means of an iterative numerical procedure with results quite similar to those by Anand et al. (2005).

A problematic aspect of the nonclassical plastic limit analysis previously described, so far not explicitly noticed, arises for certain structure geometries and loading conditions, as, typically, thin beams, or foils, in bending and thin wires in torsion. For such structures, the shape of the collapse mechanism proves to be fixed, such that the higher order boundary conditions on the (microscopically) free boundary can be enforced only by means of some sort of singularities arising in the vicinity of the boundary surface.

For example, for a thin beam in pure bending being in the plastic limit state, the plastic strain has to be linearly distributed in the (rectangular) cross section height, whereas the relevant higher order boundary conditions at the top and bottom "free" sides of the cross section demand that the slope of the plastic strain profile be there vanishing. This means that the plastic strain profile has to exhibit slope jumps at points just close to the extremes of the cross section height, where thus the curvature of the strain profile is unbounded, and this in turn – through the strengthening law – implies that the yield strength is also unbounded together with the stress. These singularities substantiate in the formation of top and bottom extremely thin rigid-plastic boundary layers sustaining infinite stresses, but finite stress resultants, which do contribute to the limit bending moment value. Similar considerations hold for a thin wire in torsion, for which the singularity consists in a slope jump of the plastic shear strain radial profile at points close to the boundary surface and thus the latter surface plays the role of thin boundary layer.

The above singularity mechanism is peculiar of rigid-plastic materials of gradient type, it thus manifests itself not only when such a material is in a state of plastic collapse, but also whenever it deforms under increasing load. Engelen et al. (2006) and Idiart et al. (2009) addressed thin foils in bending in the hypothesis of elastic–plastic hardening behavior. In analogy to the experimental study by Stölken and Evans (1998), they built a closed-form solution from where they derived the analogous solution for the limit case of rigid-plastic hardening behavior. They found that, in the latter limit case, the bending moment is formed up by, beside the standard contribution from the bulk stress, a nonstandard contribution from some top and bottom layers in the foil thickness, where the stresses are infinite, but the stress resultants are finite.

The purpose of the present paper is to address the (nonconventional) plastic limit analysis of thin foils in pure bending and of thin wires in pure torsion, and in particular to show how the limit bending and torque moments vary with the internal length scale parameter. It is found that, whereas the global strength of the specimens increases with the decreasing size, the local strength may decrease correspondingly, as it actually occurs in the central regions of the cross sections. This is in accord with the predictions of the gradient plasticity theory by Gurtin and Anand (2005) about strengthening and weakening within plastically deformed micron scale structures. It is also found that the dissipation power (power wasted as heat per unit volume) is size-independent, it thus coincides with its classical counterpart, whereas the stress power (plastic work rate done by the Cauchy stresses) is instead size-dependent.

It would be desirable to compare the obtained results with analogous experimental results, but these latter seem to be lacking, to the author's knowledge. The experiments conducted by Stölken and Evans (1998) and Moreau et al. (2005) for thin foils in bending and by Fleck et al. (1994) for thin wires in torsion were not so extensive to include the specimen's plastic collapse. Nevertheless, the results provided with the present study may constitute useful predictions for safety judgments within the domain of micron scale structures.

The essentials of the above nonclassical limit analysis theory referred to in the present study (Polizzotto, 2010a) are illustrated in Section 2 together with some additions necessary to set up a basis for the applications to thin foils in bending (Section 3) and thin wires in torsion (Section 4). For the theoretical bases of this theory, readers are requested to consult the latter quoted paper.

Notation. A compact notation is used, with boldface letters denoting vectors or tensors of any order. The scalar product between vectors or tensors is denoted with as many dots as the number of contracted index pairs. For instance, denoting by $\boldsymbol{u} = \{u_i\}$, $\boldsymbol{v} = \{v_i\}$, $\varepsilon = \{\varepsilon_{ij}\}$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}$ and $\boldsymbol{A} = \{A_{ijkh}\}$ some vectors and tensors, one can write: $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$, $\boldsymbol{\sigma} : \varepsilon = \sigma_{ij}\epsilon_{ji}$, $\boldsymbol{A} : \varepsilon = \{A_{ijkh}\varepsilon_{hk}\}$. The summation rule for repeated indices holds and the subscripts denote components with respect to an orthogonal Cartesian co-ordinate system, say $\boldsymbol{x} = (x_1, x_2, x_3)$. An upper dot over a symbol denotes its time derivative, $\dot{\boldsymbol{u}} = \partial\boldsymbol{u}/\partial t$. The symbol ∇ denotes the spatial gradient operator, i.e. $\nabla\boldsymbol{u} = \{\partial_i u_j\}$, ∇^{sym} is the symmetric part of ∇; $\overset{(n)}{\Delta} := \nabla - \mathbf{n}\partial_n$ denotes the tangential gradient on a surface element with unit normal $\mathbf{n}$. The symbol $:=$ means equality by definition. Other symbols will be defined in the text at their first appearance.

2. The plastic collapse load problem

Let us consider a solid body occupying the (open) domain V of boundary surface $S = \partial V$, which undergoes conventional small deformations under external loading actions and, in its initial undeformed state, is referred to Cartesian orthogonal co-ordinates, say $\mathbf{x} = (x_1, x_2, x_3)$. An idealized rigid-perfectly plastic material is considered, with the yield strength exhibiting size effects.

2.1. Strengthening potential and strengthening surface

Following Polizzotto (2010a), the latter size effects (referred to as "strengthening effects" in the sequel) are simulated by means of a fictitious isotropic hardening featured by a suitable hardening potential, say $\psi_{\text{st}}(\kappa, \nabla\kappa)$, where κ denotes the effective plastic strain, assumed to be C^1-continuous in V. ψ_{st}, called *strengthening potential*, has to possess the following requisites:

- $\psi_{\text{st}}(\kappa, \nabla\kappa)$ is a positively degree-one homogeneous function of its arguments, i.e. $\psi_{\text{st}}(\alpha\kappa, \alpha\nabla\kappa) = \alpha\psi_{\text{st}}(\kappa, \nabla\kappa), \quad \forall\alpha > 0.$
- $\psi_{st}(\kappa, \nabla\kappa) = 0$ whenever $\nabla\kappa = \mathbf{0}$.