European Journal of Mechanics A/Solids 57 (2016) 29–33

Contents lists available at ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

Flow rules for rigid-plastic strain kinematic hardening solids

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ARTICLE INFO

Article history:
Received 13 April 2015
Accepted 10 November 2015
Available online 2 December 2015

Keywords:
Constitutive behavior
Plastic material
Maximum plastic dissipation

ABSTRACT

The associated flow rule is commonly supposed to be one of the cornerstones of classical plasticity theory for metals though some experimental results do not accord with it. This paper investigates the flow rule of plastic deformation rate for von Mises materials exhibiting kinematic hardening. It is found that the associated flow rule is not valid in rigid-plastic deformations. The associated flow rule for von Mises materials exhibiting kinematic hardening is modified. The modified associated flow rule implies that the vector of plastic deformation rate need not be perpendicular to the yield surface in nine-dimensional stress space. Finally, the principle of maximum plastic dissipation is modified as well.

1. Introduction

The associated flow rule of plastic deformation rate and the convexity of the yield surface in stress space play an important role in the classical plasticity theory of hardening metals (e.g., Nemat-Nasser, 1983 and Khan and Huang, 1995). However, Spitzig and Richmond (1984) found that hydrostatic pressure dependence of the flow stress does not require an associated plastic volume change in metals. Besides, many researchers also questioned the validity of the associated flow rule in the absence of pressure-sensitive effect. Kuroda and Tvergaard (2001) showed that the associated flow rule is only valid at initial yield and the direction of the plastic deformation rate depends on the stress or the deformation rate when a vertex forms on the yield (loading) surface. Hashiguchi (2005) proposed a non-associated flow rule by using the sub-loading surface, the tangential yield surface and the tangential loading criterion for tangential plastic deformation rate, which describes the components of plastic deformation rate tangential to the yield surface appropriately. Stoughton, 2002, Stoughton and Yoon (2004) proposed a model for sheet metal forming in which the plastic potential and the yield functions are defined by two different quadratic functions of stress tensor respectively. The fact that experimental results do not accord with the associated flow rule prompts us to examine it theoretically. We will also investigate the principle of maximum plastic dissipation as it can lead to the associated flow rule.

2. General background

Rigid-plastic solid can be regarded as a special case of elastic–plastic solids. Loaded rigid-plastic solid can produce very small elastic deformations before it yields. However, the elastic deformation rate can be ignored compared with the plastic deformation rate in the region of elastic–plastic deformation. For rigid-plastic solid, we have

$$\mathbf{D} \approx \mathbf{D}^{\mathbf{p}} \tag{2.1}$$

where $\mathbf{D}$ denotes the deformation rate and $\mathbf{D}^{\mathbf{p}}$ the plastic deformation rate.

The deformation gradient is decomposed in the following form (cf. Naghdi, 1990 and Truesdell and Noll, 2004)

$$\mathbf{F} = \mathbf{V}\mathbf{R} \tag{2.2}$$

where $\mathbf{F}$ is the deformation gradient, $\mathbf{V}$ the left stretch tensor and $\mathbf{R}$ the rotation tensor. In the Cartesian coordinate system, the left stretch tensor is decomposed in the following form

$$\mathbf{V} = \mathbf{R}_{\mathbf{E}}\mathbf{V}_{\lambda}\mathbf{R}_{\mathbf{E}}^{\mathbf{T}} \tag{2.3}$$

where $\mathbf{V}_{\lambda}$ is a diagonal matrix whose diagonal components are the principal values of the left stretch tensor, $\mathbf{R}_{\mathbf{E}}$ is a normal orthogonal matrix. Bold letters denote tensors or tensor component matrices in the Cartesian coordinate system throughout the paper. The stress can be expressed as

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http://dx.doi.org/10.1016/j.euromechsol.2015.11.002
0997-7538/

$$\boldsymbol{\sigma} = \mathbf{R}_\mathbf{p}\boldsymbol{\sigma}_\lambda\mathbf{R}_\mathbf{p}^\mathbf{T} \tag{2.4}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\boldsymbol{\sigma}_\lambda$ is a diagonal matrix whose diagonal components are the principal values of the Cauchy stress and $\mathbf{R}_\mathbf{p}$ is a normal orthogonal matrix.

For isotropic-elastic deformation, the Cauchy stress is coaxial with the left stretch tensor. We have

$$\mathbf{R}_\mathbf{p} = \mathbf{R}_\mathbf{E} \tag{2.5}$$

Substituting (2.3) into (2.2), we obtain

$$\mathbf{F} = \mathbf{R}_\mathbf{E}\mathbf{V}_\lambda\mathbf{R}_\mathbf{L}^\mathbf{T} \tag{2.6}$$

where $\mathbf{R}_\mathbf{L}^\mathbf{T} = \mathbf{R}_\mathbf{E}^\mathbf{T}\mathbf{R}$. The deformation rate is defined as

$$\mathbf{D} = (1/2)\left[\dot{\mathbf{F}}\mathbf{F}^{-1} + \left(\dot{\mathbf{F}}\mathbf{F}^{-1}\right)^\mathbf{T}\right] \tag{2.7}$$

Substituting (2.6) into (2.7), we obtain

$$\mathbf{D} = \mathbf{R}_\mathbf{E}(\mathbf{D}_\mathbf{d} + \mathbf{D}_\mathbf{s})\mathbf{R}_\mathbf{E}^\mathbf{T} \tag{2.8}$$

where

$$\mathbf{D}_\mathbf{d} = \dot{\mathbf{V}}_\lambda\mathbf{V}_\lambda^{-1} \text{ and } \mathbf{D}_s = (1/2)\left(\mathbf{V}_\lambda^{-1}\mathbf{R}_\mathbf{L}^\mathbf{T}\dot{\mathbf{R}}_\mathbf{L}\mathbf{V}_\lambda + \mathbf{V}_\lambda\dot{\mathbf{R}}_\mathbf{L}^\mathbf{T}\mathbf{R}_\mathbf{L}\mathbf{V}_\lambda^{-1}\right) \tag{2.9a,b}$$

$\mathbf{D}_\mathbf{d}$ is a diagonal matrix, $\mathbf{D}_\mathbf{s}$ is a symmetric matrix whose diagonal components are equal to zero. $\mathbf{R}_\mathbf{L}^\mathbf{T}\dot{\mathbf{R}}_\mathbf{L}$ is a skew-symmetric matrix. When $\mathbf{V}_\lambda$ is equal to the unit matrix $\mathbf{I}$,

$$\mathbf{D}_\mathbf{s} = \mathbf{O} \tag{2.10}$$

where $\mathbf{O}$ denotes the zero matrix. When $\mathbf{V}_\lambda$ approaches $\mathbf{I}$, $\mathbf{D}_\mathbf{s}$ approaches $\mathbf{O}$ if $\mathbf{R}_\mathbf{L}$ is continuous ($\mathbf{R}_\mathbf{L}^\mathbf{T}\dot{\mathbf{R}}_\mathbf{L}$ is finite).

3. An analysis of the associated flow rule of plastic deformation rate

We consider von Mises material exhibiting kinematic hardening. The yield surface in stress space is represented by

$$F = \sqrt{(3/2)\bar{\mathbf{s}} : \bar{\mathbf{s}}} = \sqrt{(3/2)\left(s_\mathbf{ij} - \alpha_\mathbf{ij}^*\right)\left(s_\mathbf{ij} - \alpha_\mathbf{ij}^*\right)} = \sigma_{\mathbf{y}0} \tag{3.1}$$

where $\bar{\mathbf{s}} = (\mathbf{s} - \boldsymbol{\alpha})$, $\mathbf{s}$ and $\boldsymbol{\alpha}$ with components $s_\mathbf{ij}$ and $\alpha_\mathbf{ij}^*$ are the deviatoric Cauchy stress and the deviatoric back stress respectively and σ_{y0} is the equivalent Cauchy stress at initial yield. In the rectangular Cartesian coordinate system with the stress principal axes as coordinates (the stress state on the yield surface is a diagonal matrix, i.e., $\boldsymbol{\sigma} = \boldsymbol{\sigma}_\lambda$), the yield surface is rewritten as

$$F = \sqrt{(3/2)\bar{\mathbf{s}} : \bar{\mathbf{s}}} = \sqrt{(3/2)(s_i - \alpha_\mathbf{ij})(s_i - \alpha_\mathbf{ij})} = \sigma_{y0} \tag{3.2}$$

where (s_i) is a diagonal matrix. Signs with one index denote diagonal matrices throughout the paper. The deviatoric back stress at initial time may not be equal to zero (the material was subjected to plastic deformations), and thus the deviatoric back stress need not be coaxial with the stress and $(s_\mathbf{i} - \alpha_\mathbf{ij})$ may not be a diagonal matrix.

For the yield surface (3.2), the associated flow rule is

$$\mathbf{D}^\mathbf{p} = \left(D_{ij}^\mathbf{p}\right) = \dot{\Psi}\left(\frac{\partial F}{\partial \sigma_{ij}}\right)\Big|_{\sigma_{ij}=0,\ (\mathbf{i}\neq j)} = \dot{\phi}\left(s_\mathbf{i} - \alpha_\mathbf{ij}\right), \quad (i,\ j = 1,\ 2,\ 3) \tag{3.3}$$

where $\dot{\phi}$ is a scalar quantity. The associated flow rule (3.3) is also referred to as the normality flow rule as it implies that the vector of the plastic deformation rate is perpendicular to the yield surface in nine-dimensional stress ($\sigma_\mathbf{ij}$) space. The normality flow rule (3.3) is generally expressed as

$$\mathbf{D}^\mathbf{p} = \frac{1}{h}\frac{\dot{\boldsymbol{\sigma}} : \bar{\mathbf{s}}}{\bar{\mathbf{s}} : \bar{\mathbf{s}}}\bar{\mathbf{s}}, \tag{3.4}$$

where h is a scalar parameter of material.

We analyze the normality flow rule (3.4) in terms of plastic dissipation. Consider a stress state:

$$\mathbf{s} = \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & 0 \\ 0 & 0 & s_{33} \end{pmatrix}, \quad \bar{\mathbf{s}} = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & 0 & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & 0 \end{pmatrix}, \tag{3.5a,b}$$

which is on the yield surface (3.2). If the stress rate ($\dot{\boldsymbol{\sigma}}$) is directed toward the outside of the yield surface, or the inner product between the stress rate and the unit normal to the yield surface ($\dot{\boldsymbol{\sigma}} : \bar{\mathbf{s}}/\sqrt{\bar{\mathbf{s}} : \bar{\mathbf{s}}}$) is greater than zero, the normality flow rule (3.4) is supposed to be valid. From (3.4) and (3.5a,b), we obtain the dissipation of mechanical energy

$$En = \mathbf{D}^\mathbf{p} : \boldsymbol{\sigma} = \frac{1}{h}\frac{\dot{\boldsymbol{\sigma}} : \bar{\mathbf{s}}}{\bar{\mathbf{s}} : \bar{\mathbf{s}}}(s_\mathbf{ij} - \alpha_\mathbf{ij})\sigma_\mathbf{ij} = \frac{1}{h}\frac{\dot{\boldsymbol{\sigma}} : \bar{\mathbf{s}}}{\bar{\mathbf{s}} : \bar{\mathbf{s}}}(s_\mathbf{ij} - \alpha_\mathbf{ij})s_\mathbf{ij} = 0 \tag{3.6}$$

It is clear that the normality flow rule (3.4) violates the principle of plastic dissipation.

We analyze the normality flow rule (3.3) from another perspective. Consider a deformation from t_0, t_1 up to t_2: $\mathbf{I} \to \mathbf{F}_1 \to \mathbf{F}_2$. $\mathbf{I} \to \mathbf{F}_1$ is a very small isotropic-elastic deformation and $\mathbf{F}_1 \to \mathbf{F}_2$ is an elastic–plastic deformation. Assume that the elastic deformation rate can be ignored compared with the plastic deformation rate in the elastic–plastic region.

The deformation gradient is decomposed in the form (2.6). The deformation rate is decomposed in the form (2.8). We assume that the deformation gradient $\mathbf{F}$ is continuous (we does not assume that the deformation rate is continuous). Thus, $\mathbf{R}_\mathbf{L}$, $\mathbf{R}_\mathbf{E}$ and $\mathbf{V}_\lambda$ in (2.6) are continuous. The deformation gradient at t_1 is expressed as

$$\mathbf{F}_1 = \mathbf{R}_{\mathbf{E}1}\mathbf{V}_{\lambda 1}\mathbf{R}_{\mathbf{L}1}^\mathbf{T} \tag{3.7}$$

where subscript 1 denotes the time t_1. The deformation rate at t_1 is expressed as

$$\mathbf{D}_1 = \mathbf{R}_{\mathbf{E}1}(\mathbf{D}_{\mathbf{d}1} + \mathbf{D}_{\mathbf{s}1})\mathbf{R}_{\mathbf{E}1}^\mathbf{T} \tag{3.8}$$

Since $\mathbf{I} \to \mathbf{F}_1$ is a very small deformation, $\mathbf{V}_{\lambda 1}$ approaches $\mathbf{I}$ and $\mathbf{D}_{\mathbf{s}1}$ approaches $\mathbf{O}$ (see (2.10)) ($\mathbf{D}_{\mathbf{s}1}$ can be ignored compared with $\mathbf{D}_{\mathbf{d}1}$). The deformation rate at t_1 is equal to

$$\mathbf{D}_1 = \mathbf{R}_{\mathbf{E}1}\mathbf{D}_{\mathbf{d}1}\mathbf{R}_{\mathbf{E}1}^\mathbf{T} \tag{3.9}$$

The elastic deformation rate is ignored. As a result, the plastic deformation rate at t_1 is

$$\mathbf{D}_1^\mathbf{p} = \mathbf{D}_1 = \mathbf{R}_{\mathbf{E}1}\mathbf{D}_{\mathbf{d}1}\mathbf{R}_{\mathbf{E}1}^\mathbf{T} \tag{3.10}$$

Since $\mathbf{I} \to \mathbf{F}_1$ is an isotropic-elastic deformation, the stress at t_1 can be expressed as