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# Wrinkling of cylindrical membranes with non-uniform thickness

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ABSTRACT

Thin membranes are prone to wrinkling under various loading, geometric and boundary conditions, affecting their functionality. We consider a hyperelastic cylindrical membrane with non-uniform thickness pressurized by internal gas or fluid. When pre-stretched and inflated, the wrinkles are generated in a certain portion of the membrane depending on the loading medium and boundary conditions. The wrinkling is determined through a criterion based on kinematic conditions obtained from non-negativity of Cauchy principal stresses. The equilibrium solution of a wrinkled membrane is obtained by a specified combination of standard and relaxed strain energy function. The governing equations are discretized by a finite difference approach and a Newton–Raphson method is used to obtain the solution. An interesting relationship between stretch induced softening/stiffening with the wrinkling phenomenon has been discovered. The effects of pre-stretch, inflating medium, thickness variations and boundary conditions on the wrinkling patterns are clearly delineated.



## 1. Introduction

Thin elastic membranes show geometric, material and force non-linearities, which make them vulnerable to instabilities when subjected to pressure, force or traction. The stability investigation of inflated membranes is concentrated on three main topics: limit point instability, bifurcation instability and wrinkling. Limit points and bifurcation points are the two major types of critical points in the equilibrium paths. Detailed investigations of limit and bifurcation point instabilities, with some experimental results, are available in the literature (Alexander, 1971; Haughton and Ogden, 1979b, a; Pamplona et al., 2001; Eriksson and Nordmark, 2012; Patil and DasGupta, 2015; Patil et al., 2015) and references therein.

Wrinkling, resulting from compressive stresses, often occurs for inflated membrane structures, and can be seen as related to geometric effects. The wrinkles develop in the membrane in the direction orthogonal to the negative principal stress. This breaks the convexity condition of the strain energy density function of the membrane (Pipkin, 1986; Steigmann, 1990), so standard strain energy function can not be used in the wrinkled regions. Several theories regarding the wrinkling phenomenon are available in the literature (Wagner, 1929; Reissner, 1938; Wu and Canfield, 1981; Pipkin, 1986, 1993, 1994; Steigmann, 1990; Mansfield, 1970), and detailed investigations on wrinkling phenomena can be found in Li and Steigmann (1994a, b); Roxburgh (1994); Haseganu and Steigmann (1994); Massabo and Gambarotta (2007); Bonin and Seffen (2014).

In a work on wrinkling, Pipkin (1986) proposes the use of a relaxed strain energy function to study wrinkling of an isotropic membrane, with wrinkling idealized as continuously distributed over membrane surface to maintain the strain compatibility. This work shows that when a relaxed energy function is replacing the standard strain energy functions, tension field theory appears as an integral part of the membrane theory, and automatically satisfies several conditions including convexity and Legendre-Hadamard conditions. In subsequent papers, Pipkin (1993, 1994) has proved minimum energy and minimum complementary energy theorems with a relaxed strain energy density function for small and large deformation of membranes.

The wrinkling of thin elastic sheets or films are studied by either assuming thin sheets as membranes (zero bending stiffness) or as thin shells (non-zero bending stiffness). When sheets are modeled as membranes, the tension field theory is used to ascertain stress distribution and wrinkling regions (Mansfield, 1970; Pipkin, 1986; Steigmann, 1990). But, due to absence of bending stiffness, the fine structure of wrinkles is unknown in the membranes. When sheets are modeled as thin shells, the bending stiffness defines the fine structure of wrinkles (Cerda et al., 2002; Cerda and Mahadevan, 2003; Healey et al., 2013; Nayyar et al., 2011, 2014;

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Taylor et al., 2014), but it is well known that shell modeling of thin sheets is numerically demanding and the obtained results highly dependent on a used discretization. Cerda et al. (2002) presented experimental data and scaling analysis, which shows that the wrinkling wavelength decreases and wrinkling amplitude increases monotonically with increase in nominal strain for stretched elastic sheets with small finite strains. Cerda and Mahadevan (2003) extended the scaling analysis for generalized wrinkling phenomenon. However, Nayyar et al. (2011, 2014) show that for hyperelastic stretched sheets with large finite strains, wrinkling wavelength decreases with increase in nominal strain, but wrinkling amplitude increases first and then decreases with increase in nominal strain and eventually flattens out completely. Recently, Taylor et al. (2014) applied Koiter's nonlinear plate theory to stretched thin elastic sheets to obtain the wrinkling patterns.

The wrinkling of the membranes can be studied by two approaches. First, out of plane geometric non-linearities are treated as constitutive non-linearities through modification of the strain energy function, which enables to model wrinkles as continuously distributed over the membrane surface (Pipkin, 1986; Steigmann, 1990; Wagner, 1929; Reissner, 1938). A second approach is based on a modification of the deformation tensor without modifying the constitutive relationship (Wu and Canfield, 1981; Roddeman et al., 1987a, b; Lu et al., 2001). In this paper, we are following the first approach.

Steigmann (1990) proposes a general finite deformation theory of a tension field in isotropic elastic membranes. The state of membrane, whether it is tense, wrinkled or slack, is described by kinematic conditions in terms of stretches, elaborated by Pipkin (1986); Li and Steigmann (1994b, a); Roxburgh (1994). As mentioned by Massabo and Gambarotta (2007), the membrane or a part of the membrane can be in three states of stress: tense state, where the material is in a bi-axial state of stress; wrinkled state, where only one tensile principal stress is present in the direction of wrinkles; slack state, where both in-plane principal stresses are absent and the membrane is inactive. The relaxed strain energy functions for an isotropic membrane have been derived or used by Pipkin (1986) for a neo-Hookean model, by Roxburgh (1994) for a Mooney-Rivlin model, by Li and Steigmann (1994b, a) for an Ogden model, by Steigmann (2005) for a Varga model and by Massabo and Gambarotta (2007) for a Fung type model.

Even if much work has been done on wrinkling of isotropic membranes, wrinkling of anisotropic membranes remains to be explored. It was Roddeman et al. (1987a, b), who derived a general theory for wrinkled anisotropic membranes by modifying the deformation tensor. Pipkin (1994) has presented anisotropic relaxed energy functions, where the concept is based on the minimization of the energy density over all possible additions of a wrinkling strain, which is eventually found explicitly. Later, Epstein (1999) and Epstein and Forcinito (2001), show existence and uniqueness of a relaxed strain energy function for anisotropic membranes. Recently Atai and Steigmann (2014) has studied wrinkling of anisotropic sheets by dynamic relaxation method in context of modeling bio-tissues and structural membranes. The anisotropy and non-homogeneity can be used to remove a tendency of impending wrinkling (Tamadapu and DasGupta, 2013, 2014).

As the applications of membranes vary from space technologies through diverse engineering applications to biological sciences, the wrinkling instability is an unwelcome phenomenon which can be detrimental to the overall performance of membranes (Haughton and Mckay, 1997; Massabo and Gambarotta, 2007; Lu et al., 2001; Bonin and Seffen, 2014). The computational prediction of wrinkling does not always match accurately with experiments, due to many reasons like the idealization of the membrane with a specific material model, the idealization of loading and boundary conditions, a non-uniform thickness and others. Khayat et al. (1992) studied the effect of non-uniform undeformed thickness variations on the stability and deformation of cylindrical membranes. Recently, Chen (2007) presented a study on a pressurized circular membrane with linearly varying thickness, and shows that the deformation varies considerably with the thickness variation. The author notes that the micromachining process like spin coating often leads to a thicker edge and wet etching in either the thinner or the thicker edge for circular membranes. In general, for polymeric membranes non-uniform thickness of membrane occurs due to the tolerances in manufacturing process. In this paper we have studied a simple case by considering a linearly varying undeformed thickness (Chen, 2007).

To avoid further ambiguity, we define directions of wrinkles and wrinkling: the direction of wrinkles are orthogonal to the negative principal Cauchy stress and the direction of wrinkling is in the direction of negative principal Cauchy stress.

In the present work, finite inflation of a non-uniformly thin cylindrical membrane after pre-stretching is studied when subjected to fluid and gas pressures. The homogeneous, isotropic, hyperelastic membrane is modeled as a Mooney-Rivlin solid with the standard strain energy function for tense regions and a relaxed strain energy function for the wrinkled regions. Wrinkling patterns in the membrane are obtained for gas and fluid pressure loadings. Only quasi-static equilibrium solutions are considered, with dynamic and thermal effects neglected. The equilibrium equations are obtained by a variational formulation and discretized by finite differences to obtain a set of non-linear algebraic equations. The navigation between the two forms of strain energy is done with the help of a Heaviside function. The resulting non-linear algebraic equations are solved by a Newton–Raphson method. An incremental arclength-cubic extrapolation method is used to find generalized equilibrium paths.

## 2. Problem formulation

### 2.1. Kinematics of deformation

Consider an initially stress-free, homogeneous and isotropic cylindrical membrane defined by a radius $A_o$, and length $L_o$, Fig. 1. The membrane has linearly varying undeformed thickness $H$. The membrane is then subjected to a uniform axial edge load prestretching it to a new length $L_f$, where $\delta = L_f/L_o$ is an axial prestretch parameter. After reaching the targeted length $L_f$, the ends of the cylindrical membrane are fixed with rigid disks of radius $A_f$ ($A_f$ obtained from boundary conditions after pre-stretching). In general, subscript $o$ denotes parameters of the unstretched membrane and subscript $f$ those of the pre-stretched membrane. The initial radius $A_o$ and initial length $L_o$ are identical for all studied cylindrical membranes.

The undeformed configuration of the membrane is represented by the coordinates $R$, $\Theta$ and $Z$ in the radial, circumferential and axial directions, respectively. The membrane is then inflated with a fluid of density $\rho$ up to fluid level $z_w$, Fig. 1(c), or with a uniform gas pressure, Fig. 1(d). Assuming an axisymmetric configuration, the radial, circumferential and axial co-ordinates of a material point on the membrane after inflation may be represented by, respectively, $r(Z)$, $\Theta$ and $z(Z)$. The deformed radial and axial coordinates can be represented as $r(Z) = A_o + u(Z)$ and $z(Z) = Z + w(Z)$, where $u(Z)$ and $w(Z)$ are displacement field variables. The three-dimensional metric tensor for the undeformed cylindrical membrane is