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Deformation and stability of a spatial elastica under a midpoint force

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ABSTRACT

In this paper we study the deformation and stability of a pinned–pinned buckled beam under the action of a concentrated force at the midpoint. Focus is placed on the snap-through phenomenon, which may take place in a plane or three-dimensional space. We first find the equilibrium configurations by using shooting method. Elastica model is adopted to take into account exact geometry in large deformation. As expected, multiple solutions may exist for a specified set of loading parameters. Vibration method is then employed to determine the stability of the equilibrium solutions. Through these analyses the deformation sequence as the midpoint force increases quasi-statically can be predicted. It is found that the deformation sequence of the elastica is determined by two parameters; (1) the distance between the two ends of the buckled beam, and (2) the bending stiffness ratio of the cross section. Ten different deformation patterns can be identified according to four characteristics; the deformations before, after, and during the jump, and the type of critical point at the jump. A metallic wire with circular cross section is used to verify the predicted deformation sequence. It is concluded that for the specific specimen in the demonstration if one wishes to design an elastica capable of only plane deformation in all range of end distance, then the bending stiffness ratio has to be greater than 28.

1. Introduction

In the design of miniature mechanisms, such as in MEMS, it is impractical to use conventional rigid-body joint pairs due to space constraint. More often, the motion transmission is accomplished by the deformation of a single flexible beam. In particular, the snap-through phenomenon of a curved beam has been used for micro-switches (Maurini et al., 2007; Zang et al., 2007; Krylov et al., 2008; Medina et al., 2012; Ouakad and Younis, 2014). Two kinds of curved beams under midpoint forces can be found in the literature. In the first type, the curved beam is stress free when it is in curved configuration, see Fung and Kaplan (1952), Schreyer (1972), Plaut (1978), Chen et al. (2009), Virgin et al. (2014). In the second type the beam, which is stress free when it is straight, is buckled into a curved shape by edge thrust. The buckled beam is then loaded by a midpoint force, see Seide (1984), Pippard (1990), Patricio et al. (1998), Vangbo (1998), Kublanov and Bottega (1995), Pinto and Goncalves (2000), Cazottes et al. (2009), Chen and Hung (2011, 2012), Chen and Tsao (2013).

The snap-through phenomena observed in these studies are limited in plane deformation. Pippard (1990) noted in his experiment that a sway mode of instability normal to the plane of the strip tends to occur. Intuitively, if the ratio between the width and the thickness of the strip is sufficiently large, the strip should behave like a planar elastica. This raises the question what the minimum ratio is in order for the strip to deform only in a plane. Furthermore, what will happen if three-dimensional deformation does occur? In this paper we pursue this interesting phenomenon by considering an elastic rod capable of out-of-plane deformation. Elastica model is adopted in the analysis.

The elastic rod considered in this paper is stress free when it is straight. The two ends of the rod are pinned in space after it is buckled into a curved shape. A force is applied at the midpoint in the plane of the rod. The two principal moments of inertia of the cross section may be different. If the bending stiffness in one of the principal directions is much larger than the other, it is expected that the deformation will be restricted in a plane, as described in the works cited previously. If the two bending stiffness are comparable, on the other hand, spatial deformation may occur. It is the objective of this paper to identify all these deformation patterns.

In Section 2 we establish the equations of motion of the loaded rod. In Section 3 we conduct a static analysis to find the equilibrium configurations. In Section 4 we study the vibration characteristics

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of the loaded elastica. The stability of the loaded elastica can then be determined from the calculated natural frequencies. In Section 5 we present in detail a numerical example showing the load-deflection diagram and the frequency spectrum. A pinned metallic wire with circular cross section is used to verify the predicted deformation sequence as the midpoint load increases quasi-statically. In Section 6 we show a phase diagram using the distance between the two ends and the stiffness ratio between the two principal moments as two parameters. In the diagram, ten different deformation behaviors are identified. In Section 7 several conclusions are summarized.

2. Equations of motion

We consider a uniform, inextensible, and unshearable elastic rod with length L_0, cross section area A, Young's modulus E, shear modulus G, and mass density per unit volume μ. The area moment of inertia in the two principal directions of the cross section are I_1 and I_2, where $I_1 \geq I_2$. The rod is initially straight and stress free.

The rod is first buckled in the plane containing the principal direction with I_2 by pushing the two ends closer to a distance L^*. The two ends are then fixed in space with pin joints, as shown in Fig. 1(a). A space-fixed $x^*y^*z^*$-coordinate system with origin attached to the left end O is chosen to describe the geometry of the rod. The x^*-axis is pointing to the fixed end on the right. The inertial orthonormal frame $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ is associated with the $x^*y^*z^*$-coordinate system. The location of a material point on the neutral axis of the deformed rod is denoted by arc length s^* measured from end O. We consider the case when the buckled beam is loaded by point force Q^* at the midpoint $s^* = L_0/2$ in the $-y^*$ direction.

A body-fixed right-handed orthonormal frame $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ is chosen in such a manner that vector $\mathbf{d}_3$ is in the direction of the local tangent of the deformed neutral axis. Vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are in the normal cross section of the rod and rotate along with the cross section. Fig. 1(b) shows the case when the cross section is of elliptic shape with semi-major axis b and semi-minor axis a. When the rod is in the unstressed straight state, the frame $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ coincides with the frame $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ in such a manner that $\mathbf{d}_3 \equiv \mathbf{e}_x$, $\mathbf{d}_1 \equiv \mathbf{e}_y$, $\mathbf{d}_2 \equiv \mathbf{e}_z$. The vectors $\mathbf{d}_i(s^*,t^*)$ $(i = 1, 2, 3)$, where t^* is time, can be expanded as

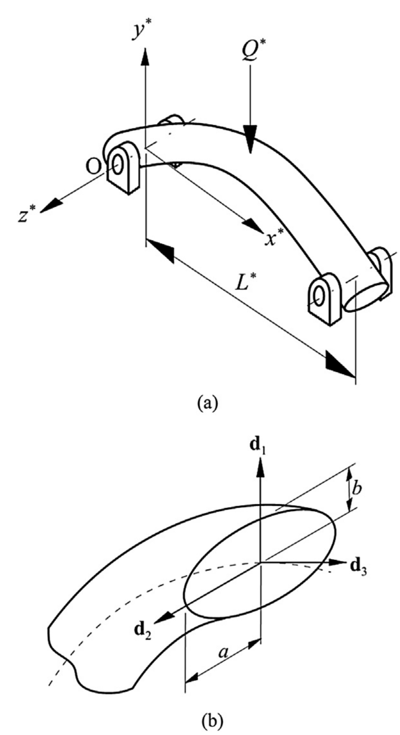

Fig. 1. (a) An elastica subject to a midpoint force. (b) Directors $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$.

$$\mathbf{d}_i = d_{ix}\mathbf{e}_x + d_{iy}\mathbf{e}_y + d_{iz}\mathbf{e}_z \quad (i = 1,\ 2,\ 3) \tag{1}$$

It is assumed that the cross section of the rod remains plane and normal to the neutral axis after deformation. The rotation of the cross section can be defined by the vector $\mathbf{d}_1$. The deformed neutral axis is a space curve defined by position vector $\mathbf{R}^*(s^*,t^*)$, which can be related to local tangent $\mathbf{d}_3$ as

$$\mathbf{R}^{*\prime} = \mathbf{d}_3 \tag{2}$$

The evolution of the frame $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ along the deformed rod is governed by the vector equation

$$\mathbf{d}_i^{*\prime} = \mathbf{\Omega}^* \times \mathbf{d}_i \quad (i = 1,\ 2,\ 3) \tag{3}$$

$()'$ in Eqs. (2)–(3) represents the derivative with respect to s^*. $\mathbf{\Omega}^*$ is the generalized strain vector (van der Heijden et al., 2003),

$$\mathbf{\Omega}^* = \kappa_1^*\mathbf{d}_1 + \kappa_2^*\mathbf{d}_2 + \tau^*\mathbf{d}_3 \tag{4}$$

κ_1^* and κ_2^* are the curvatures of projections of the neutral axis on the $\mathbf{d}_2 - \mathbf{d}_3$ and $\mathbf{d}_1 - \mathbf{d}_3$ planes, respectively. τ^* is composed of the geometric torsion of the neutral axis and the physical twist of the rod. In the case when body fixed triad $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ coincides with the Frenet trihedron of the neutral axis, then κ_1^* is zero and τ^* contains only the geometric torsion.

The internal force $\mathbf{F}^*(s^*,t^*)$ and internal moment $\mathbf{M}^*(s^*,t^*)$ can be written as

$$\mathbf{F}^* = F_1^*\mathbf{d}_1 + F_2^*\mathbf{d}_2 + F_3^*\mathbf{d}_3 = F_x^*\mathbf{e}_x + F_y^*\mathbf{e}_y + F_z^*\mathbf{e}_z \tag{5}$$

$$\mathbf{M}^* = M_1^*\mathbf{d}_1 + M_2^*\mathbf{d}_2 + M_3^*\mathbf{d}_3 = M_x^*\mathbf{e}_x + M_y^*\mathbf{e}_y + M_z^*\mathbf{e}_z \tag{6}$$

F_1^* and F_2^* are shear forces and F_3^* is the axial force. M_1^* and M_2^* are bending moments in the directions of $\mathbf{d}_1$ and $\mathbf{d}_2$, respectively. M_3^* is the twisting moment along the direction of $\mathbf{d}_3$. The constitutive equations of the rod can be written as

$$M_1^* = EI_1\kappa_1^*,\ \ M_2^* = EI_2\kappa_2^*,\ \ M_3^* = GD\tau^* \tag{7}$$

D in the third expression in Eq. (7) depends on the shape of the cross section. If the cross section is elliptic with semi-major axis a and semi-minor axis b, then $D = \frac{\pi a^3 b^3}{a^2+b^2}$. If the cross section is rectangular with sides a and b, then $D = ka^3b$, where k can be calculated from (Reismann and Pawlik, 1980)

$$k = \frac{1}{3} - \frac{192}{3\pi^5}\frac{a}{b}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\tanh\frac{(2n+1)\pi b}{2a} \tag{8}$$

The twist τ^* is zero in planar deformation, but may be nonzero in spatial deformation. By extending the formulations in (Coleman et al., 1993) and (Goriely and Tabor, 1997) from a rod with circular cross section to noncircular one, we can write the dimensionless governing equations of the spatial elastica under a midpoint force as

$$\mathbf{R}'(s,t) = \mathbf{d}_3(s,t) \tag{9}$$

$$\mathbf{F}'(s,t) - Q\delta(s-1/2)\mathbf{e}_y = \ddot{\mathbf{R}}(s,t) \tag{10}$$