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Variational formulations for the linear viscoelastic problem in the time domain

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ABSTRACT

Under the assumption of small displacements and strains, we formulate new variational principles for the linear viscoelastic hereditary problem, extending the well-known Hu-Washizu, Hellinger-Reissner, Total Potential Energy, and Complementary Energy principles related to the purely elastic problem. In addition, a new global minimum formulation is derived, giving an energetic interpretation. The new formulations are based on a convolutive bilinear form of the Stieltjes type and on the division of the time domain into two equal parts, with the resulting decomposition of the variables and of the equations governing the problem. In particular, the global minimum principle is achieved by virtue of the positive definiteness of a part of the split constitutive law operator and by means of a partial Legendre transform, and is then used to provide bounds of the overall mechanical properties of viscoelastic composite materials.

1. Introduction

The earliest variational formulations for the viscoelastic problem, although under restrictive assumptions, date back to Biot (1956), Freudenthal and Geiringer (1958), Olszak and Perzyna (1959), and Onat (1962), but the first true formulation has to be ascribed to Gurtin (1963), who generalized the classical elasticity principles to the linear viscoelastic case, using implicitly a convolutive bilinear form. Then, Gurtin, using a method able to transform initial-boundary value problems into equivalent boundary value problems, governed by integro-differential equations, formulated variational principles for the elastodynamics (Gurtin, 1964a) and for other linear initial value problems (Gurtin, 1964b). Subsequently, Tonti (1973) highlighted the crucial role played by the choice of a suitable bilinear form, in order to provide a variational formulation for the given problem. In particular, he showed how the use of a bilinear form of the convolutive type allows one to provide a variational formulation for initial value problems, making unnecessary their transformation into problems with only boundary conditions. The ideas of Gurtin and Tonti have been exploited by many authors (Schapery (1964), Leitman (1966), Taylor et al. (1970), Brilla (1972), Reddy (1976), just to name a few) and they have also been extended to the method of boundary integral equations (see Carini et al. (1991)).

Hlaváček (1966) proposed extremum formulations for isotropic viscoelastic materials with Poisson's ratio invariant in time and, under the same assumptions, a minimum formulation has been proposed also by Srinatha and Lewis (1982).

Christensen (1968, 1971), using state functions such as the free energy, proposed an extremum variational formulation, valid under restrictive assumptions. Applications of his results have been carried out by Kulejewska (1984).

Rafalski (1969, 1972, 1979) formulated extremum principles based on a bilinear form with respect to which the operator of nth derivation, with the initial conditions, proves to be self-adjoint and positive definite.

Breuer (1973) established minimum principles for incompressible viscoelastic solids. For the non-linear thermo-viscoelastic problem, new variational principles have been developed by Biot (1976).

Reiss and Haug (1978), expounding the ideas of Rafalski, formulated extremal principles for problems with initial values, including the problem of hereditary viscoelasticity.

Huet (1992), through the use of pseudo-convolutive and pseudo-biconvolutive bilinear forms, although under restrictive assumptions, obtained two principles, extensions of the minimum

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principles of Total Potential Energy and Complementary Energy related to the linear elasticity.

Tonti (1972) provided a general criterion of "potentiality", in search of variational principles. Magri (1974) illustrated the way to contrive suitable bilinear forms, with respect to which a given linear operator proves to be symmetric, and then stated a constructive method to endow any linear problem with a variational formulation. Tonti (1984) generalized the method to any non-linear problem. The latter approach was applied also to the linear viscoelasticity case (see Carini et al. (1995) and Carini and De Donato (2004)).

Many efforts have been carried out with the purpose of extending, to the viscoelastic case, the results obtained within the framework of the homogenization theory elaborated for the elastic case, with particular attention to the bounds of the "overall properties". This has been done, primarily, through quasi-elastic approximations, for special temporal values, in the domain of the Laplace transform and for methods based on complex moduli (see Hashin (1965), Schapery (1964), Minster (1973), Roscoe (1969, 1972), Chétoui (1980), Chétoui et al. (1986), Huet (1995), Cherkaev and Gibiansky (1994), Gibiansky and Milton (1993), Vinogradov and Milton (2005)). Few results have been achieved regarding the bounds of the creep and relaxation viscoelastic functions in the time domain, and only under restrictive assumptions (see Christensen (1969)) or as pseudo-elastic approximations (see Schapery (1974)). The work of Milton (1990), thanks to its generality, can be easily extended to time-dependent problems, although it has not been formulated explicitly to that purpose, leading to a variational formulation also for the viscoelasticity case. Finally, Huet (1995), using the concept of pseudo-convolutive bilinear form, derived useful unilateral and bilateral bounds for the relaxation function tensor.

In this paper we formulate five new principles: four are of the variational type and one of the minimum type. The new formulations are gathered from the decomposition of the time interval into two subintervals of equal length. Separating the variables defined over the first subinterval from those related to the second one, we obtain a formal doubling of the unknowns. Accordingly, the constitutive law operator is split into sub-operators, arranged into a two-by-two matrix that is symmetric with respect to a bilinear form of the convolutive type in the time variable. On the main diagonal, we have two operators: one is null and the other is positive definite, since the related quadratic form physically represents a free energy, positive by virtue of the results obtained in the thermodynamics field by Staverman and Schwarzl (1952a,b), Coleman (1964), Mandel (1966), Coleman and Mizel (1967), Brun (1969), Del Piero and Deseri (1996, 1997) and Amendola et al. (2012).

Nevertheless, the quadratic convolutive form associated with the whole constitutive law is not convex but, applying a partial Legendre transform, it is possible to reformulate the constitutive law so that the associated quadratic form is convex. This is a well-known technique (see Callen (1960)), used also by Cherkaev and Gibiansky (1994). The resulting minimum formulation allows one to seek bounds of the mechanical properties of a homogenized solid, given those of the viscoelastic constituents of the heterogeneous medium. Such inequalities are formally similar to those obtained by Cherkaev and Gibiansky (1994), and Milton (1990) in the frequency domain.

The paper is organized as follows. In Section 2, the linear viscoelastic problem is presented. In Section 3 and 4, respectively, the constitutive law (in the Boltzmann form) and the whole problem are rephrased on the basis of the decomposition of the time domain. Furthermore, in Section 4, the five variational formulations are provided. In Section 5, the problem is written for an RVE of a composite material made of viscoelastic phases and, in Section 6, bounds of the homogenized mechanical properties of the composite are shown. In Section 7, similar results are obtained in case the constitutive law is written in the Volterra form. Finally, in Section 8, the concluding remarks are presented.

2. The linear viscoelastic problem

Let us consider a body $\Omega \subset \mathbb{R}^3$ made of a linear viscoelastic material, that may be heterogenous and anisotropic. An orthogonal Cartesian reference system is used, with coordinates x_r , $r = 1,2,3$. The components of vectors, second order and fourth order tensors are indicated with the usual indicial notation. Einstein's convention over repeated indices is adopted.

The aim is to determine the displacement, strain and stress fields at every point of the material, for every time t in the interval $[0,2T]$, with $T > 0$, being the solid undisturbed for $t < 0$.

Let us denote by V the volume of the region Ω and by $\Gamma = \Gamma_u \cup \Gamma_p$ the external surface, with unit outward normal $n_i(x_r)$. Let $u_i(x_r,t)$, $\varepsilon_{ij}(x_r,t)$ and $\sigma_{ij}(x_r,t)$ be, respectively, the displacement, strain and stress fields at the point $x_r \in \Omega$, at the time $t \in [0,2T]$. The stress field $\sigma_{ij}(x_r,t)$, with $\sigma_{ij}(x_r,t) = 0$ for $t < 0$, satisfies the equilibrium equations:

$$\begin{aligned} &\sigma_{ij/j}(x_r,t) + b_i(x_r,t) = 0 \quad \text{in } \Omega \times [0,2T] \\ &\sigma_{ij}(x_r,t)n_j(x_r) = p_i(x_r,t) \quad \text{on } \Gamma_p \times [0,2T] \end{aligned} \tag{1}$$

where $b_i(x_r,t)$ are the volume forces, $p_i(x_r,t)$ the surface forces imposed on Γ_p, and the symbol / indicates the partial derivative operation. The displacement field $u_i(x_r,t)$ and the strain field $\varepsilon_{ij}(x_r,t)$, with $u_i(x_r,t) = 0$ and $\varepsilon_{ij}(x_r,t) = 0$ for $t < 0$, fulfill the strain-displacement relations:

$$\begin{aligned} &\varepsilon_{ij}(x_r,t) = \frac{1}{2}\left(u_{i/j}(x_r,t) + u_{j/i}(x_r,t)\right) \quad \text{in } \Omega \times [0,2T] \\ &u_i(x_r,t) = u_i^0(x_r,t) \quad \text{on } \Gamma_u \times [0,2T] \end{aligned} \tag{2}$$

where $u_i^0(x_r,t)$ is the displacement field imposed on Γ_u.

In this paper we deal only with non-aging materials (i.e., we consider only the hereditary viscoelasticity case), for which the direct constitutive law, that relates the strain field $\varepsilon_{ij}(x_r,t)$ to the stress field $\sigma_{ij}(x_r,t)$, in the Boltzmann form, reads as follows:

$$\sigma_{ij}(x_r,t) = \int_{0^-}^{t} R_{ijhk}(x_r,t-\tau)\mathrm{d}\varepsilon_{hk}(x_r,\tau) \tag{3}$$

where the integral has to be meant in the Stieltjes sense, and $R_{ijhk}(x_r,t)$, with $t > 0$ (we assume that $R_{ijhk}(x_r,t) = 0$ for $t < 0$), is the relaxation kernel. In particular, the latter is a tensor of the fourth rank whose components, functions of time and location, are obtained for a unit-step strain history. Equation (3) derives directly from the Boltzmann superposition principle (see Boltzmann (1874, 1878)), and it provides the stress field at the time t, in the fixed point $x_r \in \Omega$, due to the strain increments $\mathrm{d}\varepsilon_{ij}(x_r,\tau)$, for $\tau \in [0,t]$.

Suppose that the relaxation tensor satisfies the following symmetry properties, as in elasticity:

$$\begin{aligned} R_{ijhk}(x_r,t) &= R_{jihk}(x_r,t) = R_{ijkh}(x_r,t) \\ &= R_{hkij}(x_r,t) \quad \forall x_r \in \Omega,\ \forall t \in [0,2T] \end{aligned} \tag{4}$$

and that the following inequalities

$$R^0_{ijhk}(x_r)\gamma_{ij}\gamma_{hk} > 0, \quad R^\infty_{ijhk}(x_r)\gamma_{ij}\gamma_{hk} > 0 \tag{5}$$