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Explicit formula for amplitudes of waves in lattices with defects and sources and its application for defects detection

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ABSTRACT

A closed-form expression for the amplitudes of waves passing through the discrete uniform lattice with local defects and local sources is presented. It allows us to recover the defect properties from the available information about the amplitudes of waves. Other applications, such as modeling of cloaking devices, are also considered. The stability of this method is demonstrated by several numerical examples.

1. Introduction

In the uniform media, waves travel outward from the source in concentric circles. However, when they encounter some obstacles (defects) in their path, the new refracted and reflected waves appear and the circular symmetry is broken. Generally speaking, the defects affect the propagation of waves in the whole structure. It is natural to ask: 1) how does the wave amplitude at different points of the medium depend on the defects? 2) how to recover the properties of defects from the information about wave amplitudes at some points? 3) how to find the defect properties that model desired amplitudes of waves at fixed points of the medium? These and similar problems are related to non-destructive testing in general and appear in areas such as structural geology inversion, medical imaging, and modeling of cloaking devices. In the current research, we try to study some of these problems analytically. So, we try to find explicit formulas for the amplitudes of waves propagating in the uniform media with defects and sources and use it for explicit solutions of the direct and inverse problems mentioned above.

Recall some theoretical and mostly practical results devoted to the waves in complex media with defects, see more detailed discussions in Kutsenko (2014, 2015). Some general observations related to periodic structures without defects can be found in Brillouin (2003), Pennec et al. (2010), Torrent et al. (2013). Periodic structures with linear defects have been used in the study of waveguides, see, e.g., (Colquitt et al., 2013; Coatleven, 2012; Zhi et al., 2003; Yamada, 2011; Kim, 2010; Joseph and Craster, 2013; Korotyaeva et al., 2014). Various periodic boundaries of the media can also be considered as linear defects, see, e.g., (Karpov et al., 2005; Shuvalov et al., 2013; Kutsenko and Shuvalov, 2013; Korotyaeva et al., 2013). Local and point defects in periodic media have been treated in Maradudin (1965), Osharovich and Ayzenberg-Stepanenko (2012), Movchan and Slepyan (2007), Makwana and Craster (2013), Yao et al. (2009), Joly and Fliss (2012). The above-mentioned articles mostly have dealt with continuous media. At the same time, the discrete analogues of continuous structures are often more convenient for analysis and even have some practical advantages. For example, advanced discrete models within multi-scale methodologies (see, e.g., Karpov et al., 2005, 2007; Norris, 2014; Forest, 2009; Berezovski et al., 2009; Vernerey et al., 2007) allow us to study the properties of the media that are not available in pure continuous models. Various quantum-mechanical properties of microstructures have been studied with the help of discrete periodic operators acting on some lattices, see, e.g., (Lahiri et al., 2010; Korotyaev and Kutsenko, 2010; Badanin et al., 2013; Korotyaev and Saburova, 2014; Terrones et al., 2012). We also mention the papers Osharovich et al. (2010a, 2010b), Sohn and Krishnaswamy (2007) which study the local sources of waves in discrete media. The most popular methods of wave modeling in complex media are approximative and are based on the so-called supercell approaches (most of the above works). Defect modes can also be obtained approximately by asymptotic homogenization methods, see, e.g., (Joseph and Craster, 2013). Some analytic and

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semi-analytic algorithms involve multidimensional monodromy matrices, Dirichlet-to-Neumann operators, Green functions and explicit analysis of some algebras of multidimensional integral operators with defects, see, e.g., (Korotyaeva et al., 2014; Joly and Fliss, 2012; Martin, 2006; Kutsenko, 2014, 2015).

The current research is inspired by Rakesh and Uhlmann (2014), where the authors considered the problem of recovering smooth compactly supported potential q in the continuous equation of Schrodinger type $U_{tt}-\Delta U+qU=0$ from its far backscattering data. In the present paper, we consider the discrete wave equation $S^2U_{tt}-\Delta_{\rm discr}U+F=0$ with sources F and try to recover the slownesses S from the information about the amplitudes of waves observed at some nearby points of the lattice. For this reason, we modify and adapt the general method from Kutsenko (2014) and Kutsenko (2015) to obtain the analytic solution of the discrete wave equation with local sources and defects. Note that in contrast to Kutsenko (2014) and Kutsenko (2015) we will seek the solution for frequencies belonging to the propagating spectrum (passbands). There is no exponential decay away from the defect for the waves at these frequencies, whereas such decay is usual for localised modes in spectral gaps (stopbands). The new aspect here is also the presence of sources. It simplifies the exact analytic solution of the wave equation because we do not need to seek eigenvalues of system matrices as it must be done for the propagating, guided and localised modes without permanent sources, see Kutsenko (2014) and Kutsenko (2015). The obtained analytic solutions can be used for the analysis of inverse problems, e.g., determining of defects from the available amplitudes of waves. For example, this kind of problems arises naturally in seismic inversion (see, e.g., Virieux et al., 2012) and is usually solved approximately (and non-uniquely) by using some stochastic algorithms or some iterative descent methods.

All results are presented for two-dimensional (2D) lattices. At the same time, the method can easily be generalized to the tree-dimensional case. We note also another interesting aspect. To avoid reflection from boundaries in supercell methods, some kind of absorbing boundary conditions (e.g., PML) should be applied. In our case, there are no reflections from the artificial boundaries because we do not have them and our exact formulas model the wave propagation through an infinite lattice.

2. Core of the method

2.1. Discrete wave equation

Consider the 2D discrete uniform lattice $\mathbb{Z}^2$. At the points $\mathbf{n}\in\mathcal{N}_{\rm F}\subset\mathbb{Z}^2$ we put the sources of harmonic waves $F_{\mathbf{n}}(t)=e^{-i\omega t}f_{\mathbf{n}}$ with the same frequency ω and, possibly, different constant amplitudes $f_{\mathbf{n}}$. Consider the following equation of wave propagation in the lattice

$$\Delta_{\rm discr}U_{\mathbf{n}}(t)=S_{\mathbf{n}}^2\ddot{U}_{\mathbf{n}}(t)+\sum_{\mathbf{n}'\in\mathcal{N}_{\rm F}}F_{\mathbf{n}'}(t)\delta_{\mathbf{n}'\mathbf{n}},\quad \mathbf{n}\in\mathbb{Z}^2, \tag{1}$$

where the discrete Laplacian is

$$\Delta_{\rm discr}U_{\mathbf{n}}=\sum_{\mathbf{n}'\sim\mathbf{n}}(U_{\mathbf{n}'}-U_{\mathbf{n}}), \tag{2}$$

~ means neighboring points, δ is the Kronecker delta, $S_{\mathbf{n}}$ is the slowness at the point $\mathbf{n}$, and $U=U_{\mathbf{n}}(t)$ is the time-dependent anti-plane displacement. Assuming the time-harmonic displacements $U_{\mathbf{n}}(t)=e^{-i\omega t}u_{\mathbf{n}}$ with constant amplitudes $u_{\mathbf{n}}$ and with the same frequency ω as for the sources, we may rewrite (1) as

$$\Delta_{\rm discr}u_{\mathbf{n}}=-\omega^2S_{\mathbf{n}}^2u_{\mathbf{n}}+\sum_{\mathbf{n}'\in\mathcal{N}_{\rm F}}f_{\mathbf{n}'}\delta_{\mathbf{n}'\mathbf{n}}. \tag{3}$$

Suppose that the uniform slowness s is the same at all points of the lattice except some set of defect points $\mathcal{N}_{\rm D}\subset\mathbb{Z}^2$, i.e.

$$S_{\mathbf{n}}^2=\begin{cases}s^2, & \mathbf{n}\in\mathbb{Z}^2\backslash\mathcal{N}_{\rm D},\\ s^2+s_{\mathbf{n}}^2, & \mathbf{n}\in\mathcal{N}_{\rm D},\end{cases} \tag{4}$$

where $s_{\mathbf{n}}$ is a perturbation of the constant slowness s. Substituting (4) into (3) leads to

$$\left((\omega s)^2+\Delta_{\rm discr}\right)u_{\mathbf{n}}=-\omega^2\sum_{\mathbf{n}'\in\mathcal{N}_{\rm D}}s_{\mathbf{n}'}^2u_{\mathbf{n}}\delta_{\mathbf{n}'\mathbf{n}}+\sum_{\mathbf{n}'\in\mathcal{N}_{\rm F}}f_{\mathbf{n}'}\delta_{\mathbf{n}'\mathbf{n}}. \tag{5}$$

The linear one-to-one mapping (so-called Fourier–Floquet–Bloch transformation)

$$\mathcal{F}:\ell^2\left(\mathbb{Z}^2\right)\to L^2\left([-\pi,\pi]^2\right),\ \mathcal{F}(u_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2}=v(\mathbf{k})=\sum_{\mathbf{n}\in\mathbb{Z}^2}u_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{k}}, \tag{6}$$

allows us to rewrite the infinite linear system (5) as a functional equation on $v(\mathbf{k})$, $\mathbf{k}=(k_1,k_2)\in[-\pi,\pi]^2$. The corresponding Hilbert spaces of square-summable sequences and square-integrable functions are denoted in (6) by ℓ^2 and L^2, respectively. The inverse mapping of $\mathcal{F}$ is

$$\begin{aligned}&\mathcal{F}^{-1}:L^2\left([-\pi,\pi]^2\right)\to\ell^2\left(\mathbb{Z}^2\right),\quad \mathcal{F}^{-1}v=(u_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2},\\ &u_{\mathbf{n}}=\frac{1}{(2\pi)^2}\iint_{[-\pi,\pi]^2}e^{-i\mathbf{n}\cdot\mathbf{k}}v(\mathbf{k})d\mathbf{k},\end{aligned} \tag{7}$$

where $\mathbf{n}\cdot\mathbf{k}$ means the scalar product of vectors $\mathbf{n}$ and $\mathbf{k}$. In order to rewrite (5) in terms of the function $v(\mathbf{k})$ instead of the infinite vector $(u_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2}$, we need some properties following from (2) and (6):

$$\begin{aligned}\mathcal{F}\left((\omega s)^2u_{\mathbf{n}}+\Delta_{\rm discr}u_{\mathbf{n}}\right)_{\mathbf{n}\in\mathbb{Z}^2}&=(\omega s)^2v+\sum_{j=1,2;\ \sigma=\pm1}\left(e^{i\sigma k_j}v-v\right)\\&=Av,\end{aligned} \tag{8}$$

$$\mathcal{F}(u_{\mathbf{n}}\delta_{\mathbf{n}'\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2}=e^{i\mathbf{n}'\cdot\mathbf{k}}\left\langle ve^{-i\mathbf{n}'\cdot\mathbf{k}}\right\rangle,\quad \mathcal{F}(\delta_{\mathbf{n}'\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2}=e^{i\mathbf{n}'\cdot\mathbf{k}}, \tag{9}$$

where

$$A=(\omega s)^2-4+2\cos k_1+2\cos k_2,\ \langle\ldots\rangle=\frac{1}{(2\pi)^2}\iint_{[-\pi,\pi]^2}\ldots d\mathbf{k}. \tag{10}$$

Taking $v=\mathcal{F}(u_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2}$ and applying $\mathcal{F}$ (6) to both sides of Equation (5) and using (8)–(10) we can rewrite (5) into the equivalent form

$$Av=-\omega^2\mathbf{a}^*\mathbf{S}\langle v\mathbf{a}\rangle+\mathbf{b}^*\mathbf{f}, \tag{11}$$

where * means the Hermitian conjugation,