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Formulation of Toupin–Mindlin strain gradient theory in prolate and oblate spheroidal coordinates

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ABSTRACT

The Toupin–Mindlin strain gradient theory is reformulated in orthogonal curvilinear coordinates, and is then applied to prolate and oblate spheroidal coordinates for the first time. The basic equations, boundary conditions, the gradient of the displacement, strain and strain gradient tensors of this theory are derived in terms of physical components in these two coordinate systems, which have a potential significance for the investigation of micro-inclusion and micro-void problems. As an example, using these formulae, we formulate and discuss the boundary-value problem of a spheroidal cavity embedded in a strain gradient elastic medium subjected to uniaxial tension. In addition, the previous results given by Zhao and Pedroso (Int. J. Solids. Struct. (2008) 45, 3507–3520) in cylindrical and spherical coordinates are amended.

1. Introduction

Non-classical continuum theories have been the subject of considerable attention and study for more than one century. Historically, the idea underlying generalized continuum theories with consideration of couple stresses or body couples may be traced back to the original work of E. and F. Cosserat (1909). It was not until the 1960s that this subject was reached maturity with the works of Toupin (1962, 1964), Mindlin and Tiersten (1962), Mindlin (1964, 1965) and many other authors (e.g., Eringen, 1966; Eringen and Suhubi, 1964; Koiter, 1964; Schijve, 1966; Suhubl and Eringen, 1964). The interest in such theories lies in their intrinsic ability to introduce one or more length scales that are absent in classical continuum theories. The length scales together with the non-local nature allow for a theory that captures the size-dependence observed in many experiments, including wire torsion (e.g., Lakes, 1983, 1986; Fleck et al., 1994; Dunstan et al., 2009; Liu et al., 2013a,b; Liu et al., 2012), beam bending (e.g., Andrew and Jonathan, 2005; Haque and Saif, 2003; Hayashi et al., 2011; Kakunai et al., 1985; Lam et al., 2003; Shrotriya et al., 2003; Stölken and Evans, 1998), micro- and nano-indentation (e.g., Cordill et al., 2009; Ma et al., 2012; McElhaney et al., 1998; Nix and Gao, 1998; Stelmashenko et al., 1993; Swadener et al., 2002), etc. These size-dependent phenomena therefore further motivate the development of higher-order continuum theories (see, for example, Aifantis, 1984, 1987, 2003; Altan and Aifantis, 1997; Begley and Hutchinson, 1998; Chen and Wang, 2001; Fleck and Hutchinson, 1993, 1997, 2001; Fleck et al., 1994; Fleck and Willis, 2009a,b; Forest and Sievert, 2003; Gao and Huang, 2001; Gao et al., 1999; Gudmundson, 2004; Gurtin, 2000, 2002; Gurtin and Anand, 2005; Lam et al., 2003; Pijaudier-Cabot and Bazant, 1987; Voyiadjis and Abu Al-Rub, 2005; Yang et al., 2002; Zbib and Aifantis, 1992), especially the strain gradient theory, in which strain gradient or nonlocal terms are involved and additional material length scales are consequently introduced. An interesting review of the high-order continuum theories can be found in the works by Bazant and Jirásek (2002), Fleck and Hutchinson (1997), and Maugin and Metrikine (2010).

The range of problems being studied within the framework of strain gradient theory is extensive, including shear bands and other localizations (De Borst and Mühlhaus, 1992; Mikkelsen, 1999; Shu and Fleck, 1998; Triantafyllidis and Aifantis, 1986; Zbib and Aifantis, 1992), crack propagation (Aravas and Giannakopoulos, 2009; Chen and Wang, 2002; Huang et al., 1997; Wei and Hutchinson, 1997; Xia and Hutchinson, 1996), micro- and nano-indentation (Begley and Hutchinson, 1998; Guha et al., 2013; Nix

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and Gao, 1998; Shu and Fleck, 1998), evolution of micro-inclusion and micro-void (Bleustein, 1966; Cook and Weitsman, 1966; Chen et al., 2014; Fleck and Hutchinson, 1993, 1997, 2001; Huang and Li, 2005, 2006; Li and Huang, 2005; Li et al., 2003; Li and Steinmann, 2006; Ma and Gao, 2013; Monchiet and Bonnet, 2013; Zhang and Sharma, 2005; Zhao, 2011; Zhao et al., 2007), analysis of the static and dynamic problems (Wang et al., 2013; Xia et al., 2010; Zhang et al., 2013a,b), and many other problems. As one of the most complete higher-order continuum theories, the theory originated by Mindlin (1964) and Toupin (1962, 1964), containing four material constants (two classical and two additional) for an isotropic elastic material, has enjoyed great success. This theory has been extended to many other models. For example, Fleck and Hutchinson (1993, 1997, 2001) and Fleck et al. (1994) reformulated the Toupin–Mindlin theory and extended it to the plasticity range, in which for homogeneous isotropic and incompressible materials, the second-order deformation gradient tensor is decomposed into the stretch gradient part and the rotation gradient part. Yang et al. (2002) proposed a modified couple stress theory for elasticity by introducing the concept of the representative volume element, in which only symmetric rotation gradient tensor is considered and constitutive relations involve only one additional material length scale. Following Fleck and Hutchinson (1997), Lam et al. (2003) proposed another useful form of the strain gradient theory, namely modified strain gradient theory, which introduces three material length scales to characterize the dilatation gradient tensor, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor. In consideration of the popularity of the Toupin–Mindlin strain gradient theory, it therefore is chosen to be studied in this paper. The formulation procedure presented here can be readily extended to other higher-order continuum theories.

The original Toupin–Mindlin theory and most of other generalized continuum theories have been presented in tensorial forms, which can be recast to any specific formulations if necessary. It is straightforward to obtain the formulations in terms of rectangular coordinates for given problems where rectangular Cartesian coordinates are appropriate. However, we frequently use non-Cartesian coordinates in practical applications, especially the orthogonal curvilinear coordinates, such as cylindrical coordinates and spherical coordinates, etc. In these cases, the corresponding formulations of governing equations are always complicated, and cannot be obtained straightforwardly. Recently, Zhao and Pedroso (2008) presented the basic equations and boundary conditions of the Toupin–Mindlin theory in orthogonal curvilinear coordinates by following the procedure outlined by Eringen (2002), which was applied to cylindrical and spherical coordinates. These formulations have been used in many cases, such as cavity expansion (Bleustein, 1966; Eshel and Rosenfeld, 1970; Zhao, 2011; Zhao et al., 2007), the analysis of crack-tip fields (Aravas and Giannakopoulos, 2009; Chen et al., 1999), the thick-walled cylinder problem (Collin et al., 2009), and the buckling of carbon nanotubes (Chiroiu et al., 2010). However, most of the works are confined to the conventional cylindrical and spherical coordinates. Limited results are available for other frequently used orthogonal curvilinear coordinates, e.g., oblate and prolate spheroidal coordinates, due to their complexity. The aim of this paper is to formulate the basic equations and boundary conditions of the Toupin–Mindlin theory in the orthogonal curvilinear coordinates, with emphasis on the oblate and prolate spheroidal coordinates because they are significant for the investigation of the evolution of micro-void and micro-inclusion in various solids (see e.g., Huang and Li, 2005, 2006; Lee and Mear, 1992; Li and Huang, 2005; Li et al., 2003; Li and Steinmann, 2006; Monchiet and Bonnet, 2013; Mura et al., 1985; Ou et al., 2009a,b; Yee and Mear, 1996).

The paper is organized as follows. Basic aspects of the Toupin–Mindlin strain gradient theory are firstly reviewed in Section 2. Thereafter, the general formulations of the theory in orthogonal curvilinear coordinates are derived in Section 3. Then, expressions for the corresponding equilibrium equations, boundary conditions, and the physical components for strain and strain gradient tensors in prolate and oblate spheroidal coordinates are presented in Section 4 and Section 5, respectively. In Section 6, we formulate the boundary-value problem of a prolate spheroidal cavity embedded in a strain gradient elastic medium subjected to uniaxial tension. Finally, some discussions and the main conclusions are given in Section 7. For clarity, the symbols used are summarized as follows.

2. Review of the Toupin–Mindlin strain gradient theory

In this Section, the Toupin–Mindlin theory in rectangular Cartesian coordinates is briefly reviewed. Toupin (1962) and Mindlin (1964) developed a general isotropic strain gradient theory whereby the strain energy density depends on both the symmetric strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{1}$$

and the strain gradient tensor that is defined as the second gradient of displacement,

$$\eta_{ijk} = u_{k,ij} = \varepsilon_{ik,j} + \varepsilon_{jk,i} - \varepsilon_{ij,k}, \tag{2}$$

where $\varepsilon_{ij} = \varepsilon_{ji}$ and $\eta_{ijk} = \eta_{jik}$. This theory admits a straightforward generalization to nonlinear and inhomogeneous. It furnishes stress quantities $\sigma_{ij} (= \sigma_{ji})$ and $\tau_{ijk} (= \tau_{jik})$ which are conjugate to the generalized strain variables ε_{ij} and η_{ijk}, respectively. The work

u_i $(i = 1, 2, 3)$	Displacements
$u_{i,j}$ $(i, j = 1, 2, 3)$	Gradient of the displacement
ε_{ij} $(i, j = 1, 2, 3)$	Strain tensor components
σ_{ij} $(i, j = 1, 2, 3)$	Stress tensor components
η_{ijk} $(i, j, k = 1, 2, 3)$	Strain gradient tensor
τ_{ijk}	Higher-order stress tensor
λ, μ	Lamé constants
ξ_i $(i = 1, 2, ..., 5)$	Elastic constants associated with strain gradients
l	Material length scale
T_k $(k = 1, 2, 3)$	Surface traction components
f_k $(k = 1,2,3)$	Body forces
R_k $(k = 1, 2, 3)$	Higher-order surface traction components
$\overline{u}_i$ $(i = 1, 2, 3)$	Displacements at the kinematic surface boundary
$\overline{v}_i$ or $\overline{e}_i$ $(i = 1, 2, 3)$	Normal gradient of $\overline{u}_i$
n_i $(i = 1, 2, 3)$	Unit-normal vector
g_{ij} $(i,j = 1, 2, 3)$	Covariant components of the Euclidean metric tensor
g^{ij} $(i,j = 1, 2, 3)$	Contravariant components of the Euclidean metric tensor
g^{kk} or g_{kk}	The diagonal components of g^{ij} or g_{ij} (no sum on k)
σ_j^i and τ_k^{ij}	Mixed form of Cauchy stress and Higher-order stress
ε_j^i and η_k^{ij}	Mixed form of strain and strain gradient
$\sum_j^i$ $(i,j = 1, 2, 3)$	A generalized mixed-form second-order stress
σ_{ij}^* $(i,j = 1, 2, 3)$	Generalized stress components
A_i $(i = 1, 2, 3)$	Lamé coefficients
Γ_{ij}^k $(i,j,k = 1, 2, 3)$	The Christoffel symbols of the second kind
$u^{(k)}$, $\varepsilon_{(j)}^{(i)}$, $\eta_{(i)(j)}^{(k)}$	The physical components of u^k, ε_j^i, η_{ij}^k
δ_{ij} and δ_j^i $(i,j = 1, 2, 3)$	Covariant and mixed form of Kronecker delta
(,) at subscript	Partial differentiation
(;) at subscript	The covariant differentiation symbol
(ξ, θ, ψ)	Prolate (and oblate) spheroidal coordinates
(r, θ, z)	Cylindrical coordinates
(r, θ, φ)	Spherical coordinates