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A quadrilateral shallow shell element based on the third-order theory for functionally graded plates and shells and the inaccuracy of rule of mixtures

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A four-node quadrilateral element is developed for the dynamic analysis of doubly curved functionally graded material (FGM) shallow shells, using the refined third order theory. Two micromechanics models, the Voigt's rule of mixtures (ROM) and the Mori–Tanaka model, are considered for computing the effective material properties at a point. The accuracy of the element is examined by comparing with various three dimensional elasticity and two dimensional (2D) analytical and finite element solutions available in the literature for static and free vibration responses of FGM plates and shells. It is shown that the present element, with the least number of degrees of freedom, achieves similar or better accuracy compared to other available 2D finite elements some of which are even based on higher order theories. Using this element, we also make a systematic assessment of the accuracy of the widely used ROM in predicting the behavior of FGM structures, for different values of the inhomogeneity parameter, and different geometrical parameters, boundary conditions, and material combinations. It is revealed that there can be very significant error in the deflection, stresses and natural frequencies predicted by the ROM, depending primarily on the inhomogeneity parameter and the difference in the material properties of the constituents.

1. Introduction

Though commonly used by mother nature (e.g. in bamboo, bone, skin etc.), the concept of functionally graded materials (FGMs) as engineered materials was first introduced by Cavanagh et al. (1972), while trying to develop crack-free thermal barrier coating of gas turbine blades. The basic idea is to develop a composite material of two constituents (e.g., metal and ceramic) with a gradually varying composition from one surface to another, so that the large jumps in the inplane stresses and high interlaminar out-of-plane stresses that are induced when the materials are directly bonded to one another (e.g. in a conventional thermal barrier coating), are reduced and the resultant debonding or cracking is avoided (Kieback et al., 2003). There are a number of other advantages such as an improved distribution of residual stresses, enhanced thermal properties, higher fracture toughness, and reduced stress intensity factors (Birman and Byrd, 2007), which make them suitable for a wide number of engineering applications, such as spacecraft heat shields, nuclear power reactors, heat exchanger tubes, and biomedical implants. This is why recent years have witnessed intense research activities on the manufacturing, designing, modeling and analysis of functionally graded materials and structures.

A large body of literature is available on the theoretical modeling and analysis of beam, plate and shell-type FGM structures, recent reviews of which can be found in Birman and Byrd (2007); Jha et al. (2013a). The accuracy of these models depends on (i) the micromechanical model for the estimation of effective material properties of the FGM, and (ii) the kinematic model, i.e., the displacement field approximations across the thickness direction. In several studies, the effective properties of the FGM are directly assumed to follow a definite variation in the thickness direction, such as the exponential variation (Kashtalyan, 2004; Xu and Zhou, 2009; Vaghefi et al., 2010; Wen et al., 2011; Pendhari et al., 2012; Yang et al., 2012; Mantari and Soares, 2013; Thai and Kim, 2013; Sofiyev and Kuruoglu, 2014) and the power law variation (Yang et al., 2012). For design, it is more appropriate to specify the variation of the volume fractions of the constituents, and

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moreover, it may not be possible to physically realize certain assumed property distributions for certain material combinations (Zuiker, 1995).

For calculating the effective properties of functionally graded composites for given volume fractions and microstructure, a variety of micromechanical models have been proposed, the most common ones being the Voigt's rule of mixtures (ROM), modified rule of mixtures (MROM) (Tomoto et al., 1976), Mori–Tanaka method (MTM) (Mori and Tanaka, 1973), and the self-consistent method (Hill, 1965). These models essentially differ from each other in the degree to which they account for the interactions among the adjacent inclusions. The simplest approach, ROM, neglects this interaction altogether, and has been shown to yield very erroneous predictions of the elastic modulus. As an example, a comparison of the Young's modulus of Al/SiC composite predicted by the ROM with the experimental results of Bhattacharyya et al. (2007) is presented in Table 1, which shows an error of 38% in the ROM for a ceramic volume fraction of 40%. Still, many studies continue to use the ROM for modeling FGM structures (Reddy, 2000; Woo and Meguid, 2001; Sofiyev, 2003; Croce and Venini, 2004; Patel et al., 2005; Abrate, 2006; Zenkour, 2006; Pradyumna and Bandyopadhyay, 2008; Han et al., 2009; Tornabene et al., 2009; Akbarzadeh et al., 2011; Hamidi et al., 2012; Viola et al., 2012; Qu et al., 2013; Ebrahimi and Najafizadeh, 2013; Thai and Choi, 2013; Thai and Kim, 2013).

The Mori–Tanaka model accounts for the interaction of the elastic fields of neighboring inclusions, and yields accurate prediction for composites with a well-defined continuous matrix and a discontinuous particulate phase. This can be seen from Table 1 which shows a close agreement of the effective Young's modulus predicted by the MTM with experimental results for Al/SiC FGM. Few studies have used the MTM for obtaining the response of FGM plates and shells (Vel and Batra, 2004; Gilhooley et al., 2007; Fares et al., 2009; Mojdehi et al., 2011; Reid and Paskaramoorthy, 2011; Mantari et al., 2012; Neves et al., 2012; Taj et al., 2013; Thai and Choi, 2013; Cinefra et al., 2012). Shen and Wang (2012) presented a comparison of the Voigt and Mori–Tanaka models for the natural frequencies of FGM plates, and did not find significant difference between the two models. However, this is because the constituents of the materials chosen for the study did not have wide differences in properties. Vel and Batra (2004) employed the self consistent method also for the analysis of FGM plates, which is suitable for an interconnected skeletal microstructure without the predominance of any one phase. Kapuria et al. (2008) employed the MROM for predicting the bending and vibration response of layered FGM beams, which showed excellent agreement with experiments. The MROM, however, needs an additional fitting parameter to be determined experimentally. The experiments also established the inaccuracy of the ROM based predictions.

With regard to the kinematic modeling, many three-dimensional (3D) elasticity based solutions and two dimensional (2D) plate and shell theories have been developed for obtaining

Table 1
Effective Young's modulus Al/SiC composite.

Y (GPa)	Al/SiC ratio			
	90/10	80/20	70/10	60/40
Expt.[a]	77.1	88.5	101.6	116.8
ROM	90.5 (17.4)	114.0 (28.8)	137.5 (35.3)	161.0 (37.8)
MTM	76.5 (0.8)	87.3 (1.4)	99.7 (1.9)	114.2 (2.2)

The values in the bracket represent the error in the Young's modulus with respect to their experiential values.
[a] (Bhattacharyya et al., 2007). Materials properties: For Al, $Y = 67$ GPa, $\nu = 0.33$; For SiC, $Y = 302$ GPa, $\nu = 0.17$.

response of FGM structures. Several 3D elasticity solutions for static and dynamic responses have been presented (Vel and Batra, 2002, 2004; Kashtalyan, 2004; Xu and Zhou, 2009; Mojdehi et al., 2011; Wen et al., 2011; Pendhari et al., 2012; Yang et al., 2012), which differ from each other in terms of the distribution of effective material properties in the thickness direction, and the solution method (e.g. power series expansion, radial basis function etc.). These solutions, available for specific geometry and boundary conditions, serve as useful benchmarks for assessing the accuracy of various 2D theories.

The development of 2D models for FGM structures has been primarily directed towards extending the classical thin plate/shell theory (CPT/CST) based on Kirchhoff/Love's assumptions, which neglects shear deformation, as well as the shear deformable theories such as the first order shear deformation theory (FSDT), the refined third order theory (TOT), and the higher order theories (HOTs) from the homogeneous case to the case of varying material properties across the thickness direction. Several analytical and global approximate solutions have been presented using these theories. These include the Navier-type solution for bending of simply-supported FGM rectangular plates (Chi and Chung, 2006) and shallow spherical shells of rectangular planform (Woo and Meguid, 2001), and the Rayleigh-Ritz solution (Loy et al., 1999; Pradhana et al., 2000; Sofiyev, 2003) and the generalized differential quadrature (GDQ) solution (Ebrahimi and Najafizadeh, 2013) for free vibration and dynamic buckling of FGM cylindrical shells of arbitrary boundary conditions, using the classical shell theory. Similarly, Levy-type solution (Hashemi et al., 2011) for free vibration of FGM plates, GDQ solution for free vibration of FGM shells (Tornabene and Viola, 2013) and multisegment polynomial approximation based solution (Qu et al., 2013) for free and forced vibration of FGM shells of revolution, have been presented using the FSDT. Ferreira et al. (2005, 2006) employed the collocation multiquadric radial basis function method (RBF) to obtain the static and free vibration responses of rectangular FGM plates, and Akbarzadeh et al. (2011) and Oktem et al. (2012) presented Navier-type solutions for simply-supported FGM rectangular plates and doubly curved shells, respectively, using the TOT. Other studies using the TOT and HOTs include the GDQ solution for static response (Viola et al., 2012) and Galerkin solution (Sofiyev and Kuruoglu, 2014) for vibration and buckling response of FGM cylindrical shells. Comparisons of these 2D solutions with the 3D elasticity solutions have revealed that the absence of transverse shear deformation effect causes significant error in the CPT/CST predictions, for moderately thick and thicker FGM structures (Vel and Batra, 2004). However, the results from the FSDT and the TOT compare well with the 3D solutions.

The finite element method (FEM) is a convenient and widely used tool for obtaining solutions for practical structures of arbitrary shapes, boundary conditions and loading. Croce and Venini (2004) presented a hierarchic family of rectangular finite elements based on the FSDT for FGM plates. The FSDT, however, requires shear correction factors for the transverse shear stresses, which are arbitrary. The TOT does not need such arbitrary correction factors, but requires the interpolation function for the deflection to have C^1-continuity, which poses difficulty in developing conforming quadrilateral elements. Reddy (2000) presented conforming and nonconforming rectangular elements with eight and seven degrees of freedom (DOFs) per node based on the TOT, using the Hermite interpolation functions for the deflection, which are valid only for a rectangular shape. A similar C^1-continuous four-node rectangular plate element with ten DOFs per node has been presented by expressing the transverse displacement in terms of bending and shear components, and considering the shear term in the inplane displacement to have a (i) cubic, (ii) sinusoidal, and (iii) hyperbolic