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A unifying variational framework for stress gradient and strain gradient elasticity theories

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ABSTRACT

Stress gradient elasticity and strain gradient elasticity do constitute distinct continuum theories exhibiting mutual complementary features. This is probed by a few variational principles herein presented and discussed, which include: i) For stress gradient elasticity, a (novel) principle of minimum complementary energy and an (improved-form) principle of stationarity of the Hellinger–Reissner type; ii) For strain gradient elasticity, a (known) principle of minimum total potential energy and a (novel) principle of stationarity of the Hu–Washizu type. Additionally, the higher order boundary conditions for stress gradient elasticity, previously derived by the author (Polizzotto, *Int. J. Solids Struct.* 51, 1809–1818, (2014)) in the form of higher order boundary compatibility equations, are here revisited and reinterpreted with the aid of a discrete model of the body's boundary layer. The reasons why the latter conditions need to be relaxed for beam and plate structural models are explained.

1. Introduction

Strain gradient elasticity models of continuum mechanics are recognized within the wide literature as material models capable to capture and describe a number of experimentally detected microstructural phenomena featured by an internal length scale, such as size effects, surface effects, dispersion effects of wave propagation, along with the possibility to dispense with stress/strain singularities at, typically, crack tips and dislocation cores. Basically, the strain gradient elasticity theory is founded on the idea that the material response at a point depends not only on the local strain, but also on the strain gradients of various order, up to a value characterizing the non-simplicity grade of the material and the expected extent of physics to be captured.

The early formulations of strain gradient elasticity go back to Cauchy (1851) with his infinite series representation of an isotropic material with a crystal periodic structure, and to Voigt (1887) with his exhaustive treatment of kinematics and constitutive equations for discrete lattice models, in which the molecular displacements and rotations are considered. In the early 20*th* century the Cosserat brothers (Cosserat and Cosserat, 1909) contributed to the development of a generalized continuum theory equipped with an enriched kinematics whereby the material particle is endowed with both translational and rotational degrees of freedom. But all this remained almost unnoticed until the early 1960s, when a significant revival of gradient continuum theories occurred. The reader is addressed to Mindlin (1972) and Askes and Aifantis (2011) for further historical details on this issue. Here we only remind the landmark papers by Toupin (1962, 1964); Kröner (1963, 1967); Mindlin (1964, 1965); Mindlin and Eshel (1968); Green and Rivlin (1964a, b). Meanwhile, in the early 1970s, Germain (1973a, b) addressed the general equilibrium problem of first strain gradient materials endowed with a microstructure and inherent extra degrees of freedom. Basing on the method of virtual power, general guidelines were given which happen to unify analogous procedures emerging from the research work by Toupin (1962, 1964); Mindlin (1964, 1965, 1972); Mindlin and Eshel (1968); Green and Rivlin (1964a, b).

A second revival of gradient continuum theories took place in the early 1980s due to the work by Eringen (1983), who reformulated his earlier studies on nonlocal elasticity and devised a method whereby the original integral-type constitutive equations were replaced with differential equations of the typical form.

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1} : \left(\boldsymbol{\sigma} - \ell^2 \Delta \boldsymbol{\sigma}\right) \qquad (1)$$

where ℓ denotes an internal length scale parameter and $\mathbf{C}$ is the usual moduli tensor of isotropic elasticity. The Eringen method, although it did not provide any indications about the (higher order, (h.o.)) boundary conditions that must be associated to the latter PDEs (partial differential equations), motivated a plethora of

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applications to problems of structural mechanics analysis within micro- and nano-technologies (with the h.o. boundary conditions being heuristically devised), see e.g. (Peddieson et al., 2003; Askes and Gutiérrez, 2006; Reddy, 2007; Reddy and Pang, 2008; Kumar et al., 2008) and the literature therein, where some essential size-dependent microstructual phenomena were described. Lazar et al. (2006a) formulated a bi-Helmholtz elasticity theory based on a second stress gradient model also derived by means of the Eringen inversion procedure mentioned earlier. This model was applied to an infinite domain for dislocation analysis problems whereby no stress, nor strain, singularities were encountered.

In the early 1990s, Aifantis and his coworkers advanced a strain gradient constitutive model of isotropic elasticity (Aifantis, 1992; Altan and Aifantis, 1992; Ru and Aifantis, 1993), which may be considered as a simplified version of a more general constitutive model of strain gradient elasticity with microstructure by Mindlin (1964). The number of independent material constants of the latter Mindlin model reduces (from the huge value of 903 in the most general case) to 18 (including the Lamé constants) in the simpler case of zero relative deformations of the microstructure, and to only seven in the additional hypothesis of isotropy. In contrast, the Aifantis model requires only three constants (i.e. the Lamé constants and an internal length scale parameter, say ℓ) incorporated within an attractive diffusion-type constitutive equation as.

$$\boldsymbol{\sigma} = \mathbf{C} : \left(\boldsymbol{\varepsilon} - \ell^2 \Delta \boldsymbol{\varepsilon}\right) \tag{2}$$

The latter Aifantis model was employed to address a variety of structural problems showing that no strain singularities do occur at the crack tips and dislocation cores, and that dispersion effects in wave propagation can be effectively captured; see Askes and Aifantis (2011). However, as shown by Lazar and Maugin (2005), singularities of double stresses (that is, stresses generated by double forces (Polizzotto, 2013)) cannot be removed by means of the mentioned model.

An extension of the above first strain gradient theory to a second strain gradient one, characterized by four material constants (i.e. the Lamé constants and two length scale parameters), was advanced by Polizzotto (2003) through a constitutive equation as

$$\boldsymbol{\sigma} = \mathbf{C} : \left(\boldsymbol{\varepsilon} - \ell_1^2 \Delta \boldsymbol{\varepsilon} + \ell_2^4 \Delta\Delta \boldsymbol{\varepsilon}\right) \tag{3}$$

where ℓ_1 and ℓ_2 are length scale parameters. Lazar et al. (2006b) proposed an analogous theory cast in the form of bi-Helmholtz gradient theory and applied it to a series of dislocation problems within an infinite domain with the notable result that no singularities of any sort do arise correspondingly. The latter outcome was confirmed by further applications to defect interaction problems (Zhang et al., 2006), dislocation analysis (Lazar and Maugin, 2006; Lazar, 2013) and disclination analysis (Deng et al., 2007). However, all these applications did not pay sufficient attention to the boundary conditions, nor to the inherent surfaces effects.

Fried and Gurtin (2006) and Polizzotto (2012) provided a more extensive study on the (static and dynamic) behavior of first strain gradient elasticity models based on (2) with the additional assumption of first velocity gradient inertia. Following the guidelines prompted by Toupin (1962, 1964) and Mindlin (1964, 1965); Mindlin and Eshel (1968), the principle of virtual power was used to establish the relevant boundary conditions along with the notable boundary effects promoted by the gradient nature of the material (as the formation of a boundary layer in local and global equilibrium and the occurrence of surface inertia forces). An analogous study was undertaken by Polizzotto (2013) to address second strain gradient elasticity models based on the constitutive equation (3) under the assumption of second velocity gradient inertia. The literature concerned with the dynamic behavior of both first and second strain gradient elasticity models is extensive, see Askes and Aifantis (2006, 2011) and the references cited, but only the static features of the theory will be addressed in the following.

The mentioned Eringen method opened the way to a novel autonomous continuum theory, namely, the *stress gradient elasticity theory*, centered on the idea that the stress constitutes the driving variable within the constitutive behavior of the material and that the material response depends on the stress, as well as on the stress gradients up to some order. However, for a period as long as about thirty years after the work of Eringen (1983), models of stress gradient elasticity were obtainable only through the mentioned inversion procedure by Eringen, although discrete models were also used for this purpose (Metrikine and Askes, 2006) and homogenization methods were employed for the derivation of gradient and Cosserat elasticity models from microstructural models (Forest et al., 2001; Polizzotto, 2013). The few contributions to the formulation of such a theory are quite recent. Forest and Sab (2012) first postulated the need of formulations independent of integral ones and contributed to the construction of such a theory by addressing micromorphic elastic solids. Polizzotto (2014) elaborated a thermodynamically consistent stress gradient theory and a few related variational principles together with a stress gradient formulation of the Euler–Bernoulli beam. We shall return to the latter contributions in Section 2 with more specific comments.

A unification of the theory of Eringen with the one of Aifantis may be achieved by means of a material model in which mixed stress/strain gradient effects do take place. Such a model may perhaps be constructed by combining the constitutive equations (1) and (2) into a single one in the form.

$$\boldsymbol{\sigma} = \mathbf{C} : \mathscr{R}\left(\boldsymbol{\varepsilon} - \ell^2 \Delta \boldsymbol{\varepsilon}\right) \tag{4}$$

where $\mathscr{R}$ denotes the integral operator typical of the Eringen nonlocal continuum theories (Eringen, 1983, 2002). The constitutive model (4) exhibits the property that the material is sensitive not only to the strain and the strain gradient, but also to their respective relative mean-weighted values within the inherent domain. By the restriction that the kernel function of the integral operator be the Green function pertaining to a differential operator as $\mathrm{L}_s := 1 - \ell_s^2 \Delta$, but $\ell_s \neq \ell$, then the stress equation (4) may be inverted to produce the mixed stress/strain gradient constitutive equation advanced by Aifantis (2003), that is,

$$\boldsymbol{\sigma} - \ell_s^2 \Delta \boldsymbol{\sigma} = \mathbf{C} : \left(\boldsymbol{\varepsilon} - \ell^2 \Delta \boldsymbol{\varepsilon}\right) \tag{5}$$

Obviously, for $\ell_s = 0$ Eq. (5) coincides with the Aifantis strain gradient model (2), whereas for $\ell = 0$ it coincides with the Eringen stress gradient model (1). This mixed constitutive model has been used to address a number of crack and dislocation problems showing that both stress and strain singularities disappear (see (Aifantis, 2003) and the references therein). However, in spite of the appealing features of the latter mixed model, obtaining a rigorous thermodynamically consistent formulation of it seems to be a hard task; it will not further addressed in the present paper.

Let us also mention that a strain gradient elasticity model (together with the inherent higher order boundary conditions) can be derived from a nonlocal model by means of the principle of virtual power in conjunction with suitable Taylor series expansion techniques (Borino and Polizzotto, 2014). However, it has to be noted that Eringen (1983) proposed his stress gradient elasticity model as an alternative way to address nonlocal elasticity