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The modified version of strain gradient and couple stress theories in general curvilinear coordinates

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ABSTRACT

It is well-known that conventional continuum theory is not capable of capturing size effects and therefore, many theories have been proposed. Among these theories, strain gradient and modified strain gradient theories have enjoyed great success so far. In the present study, the modified strain gradient theory is in details derived in general curvilinear coordinates. The modified strain gradient theory involves the modified couple stress theory as a special case and therefore, the modified couple stress theory in curvilinear coordinates is readily obtained. The results are then specialized for two practical orthogonal curvilinear coordinates, i.e. cylindrical and spherical coordinates. The expressions including deformation measures, governing equations, boundary conditions and constitutive equations are expressed in terms of physical components which are more common and convenient. The formulations presented here are general and may be appropriately applied to a broad range of problems in which the use of curvilinear coordinates is unavoidable, such as problems of cylindrical and spherical cavity expansion, the analysis of asymptotic crack tip field and the interpretation of micro/nano indentation tests and also bending or twisting tests on small scales.

1. Introduction

In early 1960s, it was theoretically found that the conventional continuum theory was not general and therefore, a higher-order continuum theory was developed by some researchers such as Eringen (1962), Mindlin and Tiersten (1962) and Koiter (1964) in which concentrated moments are allowable. The so called couple stress theory introduces two new constants for an isotropic elastic material as well as Lame's constants. On the other hand, another higher-order theory was proposed by Toupin (1962) and Mindlin (1964) in which the strain energy density is considered to be a function of strain gradient in addition to classical strain. Due to involving strain gradient, the theory introduces five new constants as well as Lame's constants for an isotropic linear elastic material (Mindlin and Eshel, 1968).Mindlin (1965) also presented second strain gradient theory in which the second gradient of strain is included. Therefore, the Toupin–Mindlin theory may be obtained from Mindlin's second strain gradient theory. Since then, it has experimentally been revealed that the conventional continuum theory fails to handle problems including size effects (Lam et al., 2003; McFarland and Colton, 2005; Guo et al., 2005). Based on the two aforementioned theories, many new models have been developed (see, e.g., Bleustein, 1966, 1967; Eshel and Rosenfeld, 1970, 1975; Germain, 1973; Kang and Xi, 2007; Zhou and Li, 2001). Nevertheless, the number of new material constants makes the theories unpractical. Thus, Yang et al. (2002) imposed a higher-order equilibrium equation on the couple stress theory, the equilibrium equation of couple of couples. Consequently, the new theory only introduces one material constant, so called the material length scale parameter, in constitutive equations in addition to two classical constants. The next year, Lam et al. (2003) modified the strain gradient elasticity theory using the higher-order equilibrium equation suggested by Yang et al. (2002). The theory involves three material length scale parameters corresponding to the dilatation gradient vector, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor. It should be noted that the modified couple stress theory is a special case of the modified strain gradient elasticity theory. Strictly speaking, the modified strain gradient theory may be reduced to the modified couple stress theory if two of the three material length scale parameters are taken to be zero, i.e.

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material length scale parameters corresponding to the dilatation gradient vector and the deviatoric stretch gradient tensor.

In dealing with various applications, rectangular coordinates are selected if appropriate. However, when curvilinear coordinates are preferred, the procedure of derivation is an extremely tedious task. Thus, Zhao and Pedroso (2008) established general formulations of the Toupin–Mindlin theory in orthogonal curvilinear coordinates. Furthermore, Guzev and Qi (2013) showed how to obtain the equilibrium equations of the strain gradient theory in curvilinear coordinates. It is noteworthy that the curvilinear formulations of higher-order continuum theories are especially useful for a wide range of problems, such as problems of cylindrical and spherical cavity expansion in solids, the analysis of asymptotic crack tip field and the interpretation of micro indentation tests and also bending or twisting tests on small scales Nix and Gao (1998).In the last decade, the number of studies based on the modified version of strain gradient and couple stress theories witnesses a rising interest in employing more practical theories so that the couple stress and strain gradient theories have been replaced with the modified ones (for some recent studies, see, e.g., Ashoori Movassagh and Mahmoodi, 2013; Jomehzadeh et al., 2011; Kahrobaiyan et al., 2012; Nateghi et al., 2012; Reddy, 2011; Reddy and Kim, 2012; Sadeghi et al., 2012; Salamat-talab et al., 2013; Srinivasa and Reddy, 2013). However, formulations of the modified theories in curvilinear coordinates are absent in the literature.

The objective of the present work is to provide formulations of the modified strain gradient and couple stress theories in general curvilinear coordinates. The outline of this paper is organized as follows. Section 2 is devoted to mathematical preliminaries required here. In Section 3, the strain gradient, the modified strain gradient and the modified couple stress theories are introduced in rectangular coordinates. In Section 4, the modified version of strain gradient and couple stress theories are deduced in detail for general curvilinear coordinates. These results are simplified for orthogonal curvilinear coordinates in Section5. For more convenient, the results of foregoing section are specialized for two typical curvilinear coordinates, i.e. cylindrical and spherical coordinates, in sections 6 and 7, respectively. Finally, conclusions are drawn in section 8.

2. Strain gradient, modified strain gradient and modified couple stress theories in rectangular coordinates

In the Toupin–Mindlin strain gradient theory, the deformation measures are selected to be as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{i,j}\right) \quad \eta_{ijk} = u_{k,ij} \tag{1}$$

where u_i is the displacement vector and a comma indicates partial derivatives. Thus, the first variation of the strain energy is given by

$$\delta U = \int_V \delta\mathscr{U}\, \mathrm{d}V = \int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\eta_{ijk}\right)\mathrm{d}V \tag{2}$$

in which the tensors

$$\sigma_{ij} = \frac{\partial\mathscr{U}}{\partial\varepsilon_{ij}} \quad \tau_{ijk} = \frac{\partial\mathscr{U}}{\partial\eta_{ijk}} \tag{3}$$

are Cauchy (classical) and higher order stresses, respectively. The deformation of a strain gradient body subjected to a body force f_k is governed by the following equilibrium equation

$$\sigma_{jk,j} + \tau_{ijk,ij} + f_k = 0 \tag{4}$$

The associated boundary conditions are

$$t_k = \left(\sigma_{jk} - \tau_{ijk,i}\right) n_j + \left(n_j\partial_l^{[s]}n_l - \partial_j^{[s]}\right)\left(\tau_{ijk}n_i\right) \quad or \quad u_k = \overline{u}_k$$

$$\tilde{t}_k = \tau_{ijk}n_in_j \quad or \quad Du_k = \tilde{u}_k \tag{5}$$

The vectors t_k and $\tilde{t}_k$ represent the classical surface traction and the higher order surface traction, $\partial_i^{[s]} = \partial_i - n_iD$ and $D = n_i\partial_i$ show surface and normal gradients in which ∂_i and δ_{ij} are gradient operator and the Kronecker's delta, n_i represents unit vector normal to the surface S enclosing the body volume V. Fundamental variables are considered to be u_k and Du_k, for a vector and its surface gradient are dependent in a surface. Two prescribed functions $\overline{u}_k$ and $\tilde{u}_k$ are the displacement vector and its normal gradient on the external surface. The constitutive equations within the framework of linear elasticity are given by

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\tau_{ijk} = a_1\left(\eta_{imm}\delta_{jk} + \eta_{jmm}\delta_{ki}\right) + \frac{1}{2}a_2\left(\eta_{mmi}\delta_{jk} + 2\eta_{kmm}\delta_{ij} + \eta_{mmj}\delta_{ki}\right) + 2a_3\eta_{mmk}\delta_{ij} + 2a_4\eta_{ijk} + a_5\left(\eta_{kij} + \eta_{kji}\right) \tag{6}$$

In the preceding equations, λ and μ are Lame's constants while $a_i(i = 1, 2, \ldots, 5)$ are additional independent constants associated with gradient effects and are determined through mechanical experiments.

The modified strain gradient theory proposed by Lam et al. (2003) evolves from the Toupin–Mindlin theory. The main advantage of the modified strain gradient theory over the Toupin–Mindlin theory is the involvement of three constants. The modified theory utilizes the following four deformation measures

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{7}$$

$$\gamma_i = \varepsilon_{mm,i} \tag{8}$$

$$\eta_{ijk}^{[1]} = \eta_{ijk}^{s} - \frac{1}{5}\left(\delta_{ij}\eta_{mmk}^{s} + \delta_{jk}\eta_{mmi}^{s} + \delta_{ki}\eta_{mmj}^{s}\right) \tag{9}$$

$$\chi_{ij}^{s} = \frac{1}{4}\left(e_{imn}u_{n,mj} + e_{jmn}u_{n,mi}\right) \tag{10}$$

called the strain tensor, the dilatation gradient vector, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor, respectively and e_{ijk} stands for the permutation symbol. The tensor η_{ijk}^{s} is the fully symmetric part of the second order deformation gradient tensor given by

$$\eta_{ijk}^{s} = \frac{1}{3}\left(u_{i,jk} + u_{j,ki} + u_{k,ij}\right) \tag{11}$$

Consequently, the first variation of the deformation energy U of a body occupying a region V is

$$\delta U = \int_V \delta\mathscr{U}\, \mathrm{d}V = \int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + p_i\delta\gamma_i + \tau_{ijk}^{[1]}\delta\eta_{ijk}^{[1]} + m_{ij}\delta\chi_{ij}^{s}\right)\mathrm{d}V \tag{12}$$

The above work-conjugate to the deformation measures are defined as