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Transformed function representations of plane solutions for anisotropic elasticity and thermoelasticity

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ABSTRACT

A generalized two-dimensional deformation of an anisotropic elastic solid is considered. The transformed function method is employed to remove the breakdown limitation of the Stroh formalism for a degenerate anisotropic solid with multiple characteristic roots. The anisotropic elastic formalism for a general solution of elastic fields does not breakdown, and the closed form expressions of elastic fields for the degenerate anisotropic materials are obtained. A general solution of the thermoelastic fields in an anisotropic material under steady-state heat conduction is also derived. The thermoelastic formalism is shown to be valid for an anisotropic solid with distinct thermoelastic characteristic roots and a degenerate anisotropic solid with multiple thermoelastic characteristic roots.

1. Introduction

Two-dimensional problems in elasticity and thermoelasticity have received much attention due to their potential applications. In analyzing theoretically these two-dimensional problems, complex variable techniques have been widely used. Muskhelishvili (1953) developed a basic formulation for elastic fields in an isotropic solid under plane deformation based on the complex variable technique. Eshelby et al. (1953) and Lekhnitskii (1963) derived, in entirely different ways, a general solution for elastic fields of an anisotropic material in terms of analytic functions. Following Eshelby et al. (1953), Stroh (1958) established an anisotropic elastic formalism for a general elastic solution in anisotropic elasticity, which is referred to as the Stroh formalism. However, the Stroh and Lekhnitskii formalisms breakdown when the anisotropic material degenerates with multiple characteristic roots. Beom et al. (2012) introduced new transformed functions to remove the limitation of breakdown for an orthotropic material. They showed that the orthotropic elastic formalism based on the transformed function method recovers the classical solutions for isotropic material and degenerate orthotropic material. On the other hand, some progress for the formulations of complex function representations for thermoelastic fields in anisotropic thermoelasticity has been made. Bogdanoff (1954) and Clements (1973) derived basic formulations for thermoelastic fields in isotropic and anisotropic solids, respectively. Hwu (1990) obtained general representations for thermoelastic fields in anisotropic thermoelasticity based on the Stroh and Lekhnitskii formalisms. His formalism is not valid for a degenerate anisotropic material with multiple thermoelastic characteristic roots. Recently, Beom (2013) modified the anisotropic thermoelastic formalism for the in-plane problem, which does not breakdown for a degenerate anisotropic solid. The breakdown limitation in the anisotropic thermoelastic formalism for degenerate anisotropic thermoelastic material in the more general case of generalized two-dimensional deformation has not yet been resolved explicitly.

The purpose of this study is to investigate an elastic formalism for a general solution of elastic fields in an anisotropic elastic solid. Generalized two-dimensional deformations of the anisotropic elastic solid under plane stress and plane strain conditions were considered. The transformed function method was employed to overcome the limitation of breakdown for the degenerate anisotropic material in the Stroh formalism. The modified elastic formalism for anisotropic material was verified not to breakdown for the degenerate anisotropic material. Based on the modified elastic formalism, the dependence of stresses on anisotropic elastic constants is discussed. A general solution of thermoelastic fields in anisotropic thermoelastic material under steady state heat conduction was also derived. The thermoelastic formalism is valid in the sense of the limit for thermoelastic material with multiple elastic characteristic roots, and an anisotropic thermoelastic solid with distinct thermoelastic characteristic roots. Closed form

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expressions describing thermoelastic fields for the degenerate anisotropic material were obtained.

2. Transformed function method for anisotropic elasticity

Consider a generalized two-dimensional deformation of an anisotropic elastic solid. The three components of displacement depend only on the in-plane coordinates x_1 and x_2. The constitutive equation of a generally anisotropic material can be written in the following compact form (Lekhnitskii, 1963):

$$\varepsilon_i = \sum_{j=1}^{6} S_{ij}\sigma_j, (i = 1,2,3,4,5,6) \tag{1}$$

in which S_{ij} is the conventional compliance, and $\{\varepsilon_i\} = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ 2\varepsilon_{23}\ 2\varepsilon_{31}\ 2\varepsilon_{12}]^T$ and $\{\sigma_j\} = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{31}\ \sigma_{12}]^T$ where ε_{ij} and σ_{ij} are the strain and stress, respectively. The superscript T indicates the transpose. According to Eshelby et al. (1953), Stroh (1958), and Lekhnitskii (1963), a general solution of two-dimensional elastic fields that satisfies the equilibrium equation can be written in the following forms:

$$\begin{aligned}
u_i &= 2\mathrm{Re}\left[\sum_{j=1}^{3} A_{ij} f_j(z_j)\right], \\
\tau_i &= -2\mathrm{Re}\left[\sum_{j=1}^{3} B_{ij} f_j(z_j)\right] \\
\sigma_{1i} &= -2\mathrm{Re}\left[\sum_{j=1}^{3} B_{ij} p_j f_j'(z_j)\right], \\
\sigma_{2i} &= 2\mathrm{Re}\left[\sum_{j=1}^{3} B_{ij} f_j'(z_j)\right] \quad (i = 1,2,3).
\end{aligned} \tag{2}$$

Here, u_i and τ_i are the displacement and resultant force, respectively. Re denotes the real part and prime ($'$) designates the derivative with respect to the associate argument. $f_j(z)$ ($j = 1, 2, 3$) are complex functions, and $z_j = x_1 + p_j x_2$ where p_j ($j = 1, 2, 3$) are the roots with a positive imaginary part that satisfy the following characteristic equation for plane stress deformation

$$N\left(p_j\right) = 0 \quad (j = 1,2,3) \tag{3}$$

where

$$N(p) = \ell_2(p)\ell_4(p) - [\ell_3(p)]^2, \tag{4}$$

$$\begin{aligned}
\ell_2(p) &= S_{55}p^2 - 2S_{45}p + S_{44}, \\
\ell_3(p) &= S_{15}p^3 - (S_{14}+S_{56})p^2 + (S_{25}+S_{46})p - S_{24}, \\
\ell_4(p) &= S_{11}p^4 - 2S_{16}p^3 + (2S_{12}+S_{66})p^2 - 2S_{26}p + S_{22}.
\end{aligned} \tag{5}$$

The matrices **A** and **B** for anisotropic material under plane stress deformation are given by Stroh (1958) and Suo (1990)

$$\begin{aligned}
A_{ij} &= A_i\left(p_j\right), \\
A_{i3} &= A_i^*(p_3), (i = 1,2,3; j = 1,2),
\end{aligned} \tag{6}$$

$$\mathbf{B} = \begin{bmatrix} -p_1 & -p_2 & -\xi_3 p_3 \\ 1 & 1 & \xi_3 \\ -\eta_1 & -\eta_2 & -1 \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned}
A_1(p) &= S_{11}p^2 + S_{12} - S_{16}p + \eta(p)(S_{15}p - S_{14}), \\
A_2(p) &= S_{21}p + \frac{S_{22}}{p} - S_{26} + \eta(p)\left[S_{25} - \frac{S_{24}}{p}\right], \\
A_3(p) &= S_{41}p + \frac{S_{42}}{p} - S_{46} + \eta(p)\left[S_{45} - \frac{S_{44}}{p}\right],
\end{aligned} \tag{8}$$

$$\begin{aligned}
A_1^*(p) &= \xi(p)(S_{11}p^2 + S_{12} - S_{16}p) + S_{15}p - S_{14}, \\
A_2^*(p) &= \xi(p)\left[S_{21}p + \frac{S_{22}}{p} - S_{26}\right] + S_{25} - \frac{S_{24}}{p}, \\
A_3^*(p) &= \xi(p)\left[S_{41}p + \frac{S_{42}}{p} - S_{46}\right] + S_{45} - \frac{S_{44}}{p},
\end{aligned} \tag{9}$$

$$\begin{aligned}
\xi(p) &= -\frac{\ell_3(p)}{\ell_4(p)}, \\
\eta(p) &= -\frac{\ell_3(p)}{\ell_2(p)},
\end{aligned} \tag{10}$$

$$\begin{aligned}
\xi_3 &= \xi(p_3), \\
\eta_j &= \eta(p_j) \quad (j = 1,2),
\end{aligned} \tag{11}$$

The basic formalism given by Eq. (2) is referred to as the Stroh formalism or the LES representation (Lekhnitskii, 1963; Eshelby et al., 1953; Stroh, 1958). We consider here an anisotropic solid under plane stress deformation. For plane strain deformation, S_{ij} is replaced with S_{ij}^e, which is defined as

$$S_{ij}^e = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}. \tag{12}$$

The Stroh formalism given by Eq. (2) applies only to anisotropic material with distinct characteristic roots p_j ($j = 1, 2, 3$). When the anisotropic material degenerates to have multiple characteristic roots, the Stroh formalism breaks down. The limits of matrices **A** and **B** for the multiple characteristic roots exist even though their determinants vanish. The Stroh formalism, however, is not reduced to a classical solution for degenerate anisotropic material with multiple characteristic roots. This implies that $f_j(z)$ ($j = 1, 2, 3$) for the degenerate anisotropic material does not exist. Therefore, a modification of the Stroh formalism is needed for a degenerate anisotropic solid. Recently, Beom et al. (2012) introduced new transformed functions to overcome the limitation of breakdown for an orthotropic material, which is referred to as the transformed function method. Their method obtains a modified Stroh formalism for anisotropic material under generalized two-dimensional deformation. We define transformed functions $g_i(z)$ ($i = 1, 2, 3$) as

$$g_i(z) = \sum_{j=1}^{3} B_{ij} f_j(z), \tag{13}$$

in which $z = x_1 + px_2$, where p is a complex number with a positive imaginary part. We note from Eq. (13) that the functions $f_i(z)$ ($i = 1, 2, 3$) are written in terms of the functions $g_j(z)$ ($j = 1, 2, 3$) as

$$f_i(z) = \sum_{j=1}^{3} B_{ij}^{-1} g_j(z), \tag{14}$$

where

$$\mathbf{B}^{-1} = \frac{1}{\Delta}\begin{bmatrix} 1-\xi_3\eta_2 & p_2 - \xi_3\eta_2 p_3 & \xi_3(p_2 - p_3) \\ -1+\xi_3\eta_1 & -p_1 + \xi_3\eta_1 p_3 & \xi_3(p_3 - p_1) \\ \eta_2 - \eta_1 & \eta_2 p_1 - \eta_1 p_2 & p_1 - p_2 \end{bmatrix}, \tag{15}$$