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Fractional time-dependent deformation component models for characterizing viscoelastic Poisson's ratio

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ABSTRACT

There is the complex relationship between Poisson's ratio and time in viscoelastic solids. Hence, using a simple model to characterize accurately the viscoelastic Poisson's ratio is important for the analysis of viscoelastic behaviors. We put forward two deformation elements, fractional and spring deformation element, and obtain three fractional deformation models through connecting the two deformation elements in parallel or series. The functions of viscoelastic Poisson's ratio are also derived for stress relaxation and constant-longitudinal-strain loading. Further comparisons between tests and fitting results reveal that the fractional deformation models can represent reasonably the viscoelastic Poisson's ratio.

1. Introduction

Poisson's ratio of viscoelastic materials may be defined in several ways, and the most commonly used one is to consider it as the ratio of time-dependent transverse to longitudinal strain in axial extension or compression. Poisson's ratio in linear viscoelasticity is associated with time-dependent stress and deformation and is one of the key factors of determining numerical simulation accuracy. For example, stress in the vicinity of a bonded joint between dissimilar materials is sensitive to Poisson's ratio (Adams and Peppiatt, 1973). Hence, it is important to describe exactly the Poisson's ratio of viscoelastic materials.

Determinations of the viscoelastic Poisson's ratio may be direct or indirect. Indirect determinations involve calculation from two other time or frequency-dependent material functions, short circuiting any measurements of the transverse strain as such. But it has been found by Tschoegl et al. (2002) that indirect determinations of Poisson's ratio have hitherto been singularly unsuccessful. In direct determination, Poisson's ratio is got from actual measurements of the transverse strain. Recently, measurement techniques, which used to be the largest conundrum for the direct determination, have made significant progress (Fathi et al., 2012; Kim et al., 2003; Le Rouzic et al., 2012; Righetti et al., 2004; Wong et al., 2000). For instance, the high-definition digital camera was used to measure transverse strain (Addiego et al., 2006), which can improve the observation accuracy. However, there is much complex relationship between Poisson's ratio and time even when the viscoelastic rod is undergoing a simple uniaxial loading, for example, during the stress relaxation of compression, the Poisson's ratio of some materials is decreasing with time, while it increases over time in tensile stress relaxation of viscoelastic specimens (Colucci et al., 1997; Wong et al., 2000), and the viscoelastic Poisson's ratio need not increase with time and it also need not be monotonic with time (Lakes and Wineman, 2006). In this circumstance, using a simple model to characterize accurately the viscoelastic Poisson's ratio is essential for the analysis of viscoelastic behaviors.

Fractional calculus is an excellent mathematical instrument for modeling viscoelastic behaviors and particularly suited for building the time-dependent constitutive model. The use of fractional calculus is motivated in large part by the fact that fewer parameters are needed to achieve accurate approximation of experimental data. Up until now, it has received tremendous success in the description of the stress–strain relationship of viscoelastic materials (EL-Shahed, 2006; Hyder Ali Muttaqi Shah and Qi, 2010; Qi and Xu, 2007; Rossikhin and Shitikova, 2012; Wang and Xu, 2009), non-Newtonian fluids (Mahmood et al., 2009), anomalous diffusion (Sun et al., 2009) and other fields (Gaul et al., 1989; Kovacic and Zukovic, 2012; Lazopoulos, 2006; Ngueuteu and Woafo; Youssef

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and Al-Lehaibi, 2010). It is known that fractional calculus is employed is being used in both solid mechanics (Zhou et al., 2011) and fluid mechanics (Mahmood et al., 2009). In the application for viscoelastic materials, the fractional calculus methodology is mainly based upon the utilization of an element termed a 'spring-pot'. This element essentially replaces the dashpot in the classical Kelvin and Maxwell viscoelastic models. It is well known that ideal solids (spring) obeys Hooke's law, $\sigma(t) - \varepsilon(t)$, and Newtonian fluids (dashpot) satisfy Newton's law of viscosity, $\sigma(t) - d^1\varepsilon(t)/dt^1$, where σ is stress and ε is strain. If Hooke's law is written as $\sigma(t) - d^0\varepsilon(t)/dt^0$, it is not difficult to imagine that the spring-pot, which represents an "intermediate" material, has a fractional relation (Smit and de Vries, 1970):

$$\sigma(t) = \rho\xi^{\chi}\frac{\mathrm{d}^{\chi}\varepsilon(t)}{\mathrm{d}t^{\chi}}, \quad (0 \le \chi \le 1) \tag{1}$$

where χ may be a non-integer and $\mathrm{d}^{\chi}\varepsilon(t)/\mathrm{d}t^{\chi}$ is a fractional derivative of strain versus time. Moreover, ρ and ξ are material constants and t denotes time. Comparison with the traditional spring-dashpot model, the fractional stress–strain component model enjoys the advantage of having fewer parameters and simple forms (Schiessel et al., 1995). In the theory of viscoelasticity, the transverse–longitudinal strain relationship is as important as the stress–strain relationship. However, until now it is still unclear that whether fractional calculus can be employed to characterize the transverse–longitudinal strain relationship or the viscoelastic Poisson's ratio.

In this paper, to explore a new method for describing the viscoelastic Poisson's ratio, we are therefore intended to propose fractional time-dependent deformation component models by imitating the fractional stress–strain component models.

2. Riemann–Liouville fractional calculus

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation operator and the integration operator. There are several different definitions in fractional calculus. Here, the Riemann–Liouville fractional calculus will be introduced because it is used to obtain the relation between longitudinal strain and transverse strain.

Assuming that $f(t) = 0$ for $t = 0$, the Riemann–Liouville fractional order integral is,

$$\frac{\mathrm{d}^{-\beta}f(t)}{\mathrm{d}t^{-\beta}} = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau, \quad t>0, \ \beta\in R^{+} \tag{2}$$

where Γ is the Gamma function.

The Riemann–Liouville differential operator of order β ($\beta > 0$) is defined as a compound of integral operator of order $n - \beta$ for ($n - 1 < \beta \le n$) and n-th derivative operator, that is:

$$\frac{\mathrm{d}^{\beta}f(t)}{\mathrm{d}t^{\beta}} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{\mathrm{d}^{-(n-\beta)}f(t)}{\mathrm{d}t^{-(n-\beta)}}\right], \quad \beta>0, \ n-1<\beta\le n. \tag{3}$$

Assuming that the following function has fractional integral and derivative, fractional calculus has following primary property:

$$\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\left[\frac{\mathrm{d}^{-\beta}f(t)}{\mathrm{d}t^{-\beta}}\right] = \frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\left[\frac{\mathrm{d}^{\beta}f(t)}{\mathrm{d}t^{\beta}}\right] = f(t), \quad \beta>0. \tag{4}$$

For $f(t) = t^{\gamma}$, Eqs. (2) and (3) can be rewritten as

(a) Spring deformation element (b) Fractional deformation element

Fig. 1. Single elements.

$$\begin{cases} \frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\beta)}t^{\gamma+\beta} \\ \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)}t^{\gamma-\beta} \end{cases}, \quad \beta>0, \ \gamma>-1, \ t>0. \tag{5}$$

3. Fractional time-dependent deformation component models

3.1. The single fractional element

It can be known from (Wong et al., 2000) that the transverse strain–time curve has the similar shape with the stress–time plot during the stress relaxation in unconfined compression. We know that the stress–time experimental response can be described well by the fractional stress–strain component model. Thus, it is natural for us to expect characterize the transverse strain using a similarity method.

It is well known that the Hooke's law, $\sigma = E\varepsilon$, corresponds to $\varepsilon' = -\nu\varepsilon$ in ideal elastic solids, where ε' is the transverse strain, E and ν denote the elastic modulus and Poisson ratio. The analogy is an important researching method in science, and it has many applications in physics research and finding out new physical law. Thus, base on analogy with Eq. (1), it isn't difficult to imagine that the paired equations can be written as

$$\varepsilon' = -\eta\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}}. \tag{6}$$

We want to point out that, although Eq. (6) may be expressed as $\varepsilon' = -\eta\varepsilon$ for $\alpha = 0$, η can't be regarded as Poisson's ratio.

It is well known that the deformation in ideal elastic solid is from inner energy and the Poisson's ratio is a constant. However, the viscoelastic materials are deformed by the combined effects of internal energy and entropy (Roylance, 2001), which is the underlying physical reason for the complicated deformation and the complex viscoelastic Poisson's ratio function. As a result of entropic involvement, the molecules in viscoelastic materials can be rearranged to cause some surprising deformation. For example, after an initial "hydrostatic stress" induced volume change (increase in tension, decrease in compression), a time dependent densification was observed. The tension result is not surprising, owing to the expectation that the volume should relax as the stress relaxes. The inverse behavior found in compression is surprising (Colucci et al., 1997).

To more accurately depict the time-dependent Poisson's ratio using a simple model, imitating the fractional spring-pot element and spring element, we assume that there are two deformation elements, fractional deformation element and spring deformation element, shown in Fig. 1. The two deformation elements obey the fractional longitudinal–transverse strain relationship, Eq. (6), and $\varepsilon' = -\nu\varepsilon$ in ideal elastic solids, respectively. Analogously with the fractional Maxwell, Kelvin–Voigt and standard linear solid model, a series of deformation component models can therefore be established.

3.2. The fractional deformation Maxwell model

The standard Maxwell model is composed of a spring and a dashpot arranged in series. We generalize this model by replacing