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Analysis of a gradient-elastic beam on Winkler foundation and applications to nano-structure modelling

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ABSTRACT

An equation of motion governing the response of a first strain gradient beam, including the effect of a Winkler elastic foundation, is derived from the Hamilton–Lagrange principle. The model is based on Mindlin's gradient elasticity theory, while the Euler-Bernoulli assumption for slender beams is adopted. Higher-continuity Hermite Finite Elements are presented for the numerical solution of related Initial-Boundary Value (IBV) problems. In the static case an analytical solution is derived and the convergence characteristics of the proposed Finite Element formulation are validated against the exact response of the configuration. Several examples are presented using "equivalent beam" data for Carbon Nanotubes (CNT's) and the effect on the Winkler foundation is studied. Finally, applicability of the derived model for the simulation of micro-structures, as for example CNT's or Microtubules, is discussed.

1. Introduction

Non-classical continuum field theories have drawn much attention for more than one century due to their ability to model the micro – structural behaviour of materials (Mindlin, 1964, 1965; Mindlin and Eshel, 1968). More precisely, classical elasticity theory has been found unable to simulate adequately mechanical phenomena of theoretical, structural or technological importance (Mindlin, 1964, 1965; Mindlin and Eshel, 1968; Fried and Gurtin, 2006; Fleck and Hutchinson, 1993). This incapability becomes more obvious in the case of materials with intense micro – structure such as granular materials, foams, various polymers, polycrystals, etc. The existing literature on the subject is quite extended and covers many aspects of non – classical field theories of both theoretical and practical importance (Mindlin, 1964, 1965; Mindlin and Eshel, 1968; Fried and Gurtin, 2006; Fleck and Hutchinson, 1993; Ru and Aifantis, 1993; Altan and Aifantis, 1997; Lazar and Maugin, 2006; Polizzotto, 2012, 2014; Yang et al., 2002; Georgiadis and Grentzelou, 2006). Applications of gradient theories include elastic and plastic deformation of solids, micro-flows and others with a wide range of applications e.g. (Mindlin, 1964, 1965; Mindlin and Eshel, 1968; Fried and Gurtin, 2006; Fleck and Hutchinson, 1993).

Along with the technological advance, the need for more sophisticated equipments (micro and nano – devices) becomes a necessity. This fact has moved non – classical field theories from theory to practice, as they are now used for the design and behavior prediction of structural components (Papargyri – Beskou et al., 2003; Tsamasphyros et al., 2007; Ma et al., 2008; Reddy and Pang, 2008; Giannakopoulos and Stamoulis, 2007; Lazopoulos and Lazopoulos, 2010). The simple bending of such beams has been examined by several authors. We mention here Altan and Aifantis (1997), who considered the problem of beam vibration, Papargyri – Beskou et al. (2003), who have performed static analysis of such beams, along with the examination of buckling phenomena and finally Giannakopoulos and Stamoulis (2007), in a quite recent paper on structural gradient elastic components. Furthermore, some results on the wave dispersion relations of gradient elastic beams have been derived by Papargyri-Beskou et al. (2009).

On the other hand, the invention of carbon nanotubes (Iijima, 1991) starts a new research topic for the study of nanostructures. Recently there was an increasing interest for the applications of the nonlocal continuum theories at the specific area of nanotechnology.

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Carbon Nanotubes (CNTs) seem to pose excellent mechanical, electric and thermal properties in comparison with the conventional materials. Because of these properties, CNTs are mainly used in the fields of nano-devices and nano-electronics (Li and Chou, 2004).

In this paper we analyse the bending behavior of a strain gradient Euler – Bernoulli beam supported on a linear elastic Winkler foundation (Winkler, 1867). A sixth order differential equation is derived along with appropriate boundary conditions. The resulting IBVp has been solved numerically using a conforming C^2 continuous Finite element Method (FEM). In the present approach, the theories presented by Yang et al. (2002) and Papargyri – Beskou et al. (2003) have been combined and the effects of an elastic Winkler type support have been also included. The model features not only the derivatives of higher order in the equations of equilibrium but also terms depended on the section of the beam, which increase the rigidity of the structure. These terms have also been noticed by Ma et al. (2008), whose study has also been applied in different bending problems.

The paper is organised as follows. The derivation of the model from Hamilton's principle is presented in the following section. Section 3 deals with the wave dispersion characteristics of the beam equation. The proposed finite element procedure is analysed in Section 4. Results regarding the static and transient analysis of gradient-elastic beams on elastic foundations are studied in Sections 5 and 6, respectively and a short discussion along with the conclusions of the present study follows in Section 7.

2. Derivation of the model

We assume an elastic slender structure, resting on a Winkler foundation and the coordinate system (O, x_1, x_2, x_3), where axis x_1 coincides with the reference fibre as depicted in Fig. 1.

Due to the slenderness of the structure and invoking the Euler–Bernoulli kinematic hypothesis, it is

$$u_3 = u_3(x_1,t),\; u_2 = 0,\; u_1(x_1,x_3,t) = -x_3\frac{\partial u_3}{\partial x_1}, \tag{1}$$

where u_i, $i = 1,2,3$ is the displacement field. According to Form II of Mindlin's strain gradient theory (Mindlin, 1964), where apart from the classical strain field ε_{ij}, the gradients $\kappa_{ijk}:\ =\varepsilon_{kj,i}$ (with the standard indicial notation) are introduced, the in-plane components of the strain and strain gradient are

$$\begin{aligned}\varepsilon_{11}(x_1,x_3,t) &= -x_3\frac{\partial^2 u_3}{\partial x_1^2},\\ \kappa_{111}(x_1,x_3,t) &= -x_3\frac{\partial^3 u_3}{\partial x_1^3},\\ \kappa_{311}(x_1,x_3,t) &= -\frac{\partial^2 u_3}{\partial x_1^2}.\end{aligned} \tag{2}$$

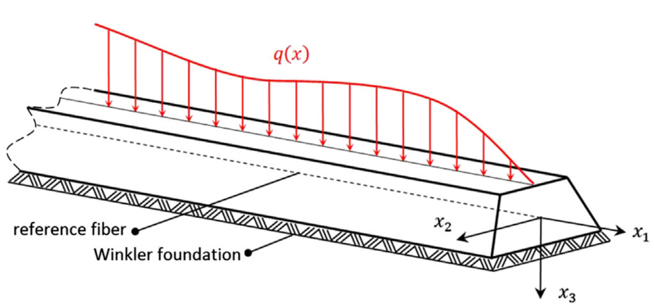

Fig. 1. A Prismatic beam on an elastic foundation.

The constitutive equations for an isotropic gradient elastic material, simplified so as to introduce only one microstructural parameter, read (Mindlin, 1964)

$$\tau_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{3}$$

$$\mu_{ijk} = g^2\left(2\mu\kappa_{ijk} + \lambda\kappa_{irr}\delta_{jk}\right), \tag{4}$$

where τ_{ij}, μ_{ijk} are the classical Cauchy and dipolar stress respectively, δ_{ij} is the Kronecker delta and g is a micro-structural constant with dimensions of length, related to the volumetric strain energy. The Lamè constants are denoted as λ, μ. In the case examined, the constitutive equations for the given stresses, using Eqs. (2)–(4), as well as, plane stress conditions on both x_1,x_2 and x_1,x_3 planes, become

$$\begin{aligned}&\tau_{11}(x_1,x_3,t) = E\varepsilon_{11},\\ &\mu_{111}(x_1,x_3,t) = g^2E\kappa_{111} \text{ and}\\ &\mu_{311}(x_1,t) = g^2E\kappa_{311},\end{aligned} \tag{5}$$

where E is Young's modulus. The strain energy $\widehat{U}$ of the considered structure is

$$U^{\frown} = \frac{1}{2}\iiint_V\left(\tau_{ij}\varepsilon_{ij} + \mu_{ijk}\kappa_{ijk}\right)dV \;+\; \frac{1}{2}\int_L k_w u_3^2 dL, \tag{6}$$

where k_w is the stiffness of the elastic foundation and L the length of the beam. Denoting by A the cross-section area of the beam, using Eqs. (2) and (5) and taking into account that $u_{3,11}$ and $u_{3,111}$ are functions of x_1 and t only, the strain energy may be written as

$$\widehat{U} = \frac{1}{2}\int_0^L\left(Cu_{3,11}^2 + Su_{3,111}^2 + k_w u_3^2\right)dx_1, \tag{7}$$

$$\text{where, } C = E(I_{11} + g^2A),\; S = g^2EI_{11} \text{ and } I_{11} = \iint_A x_3^2 dA. \tag{8}$$

Under the assumptions adopted for the type of the microstructure, namely Mindlin's strain gradient elasticity, the kinetic energy of the solid $\widehat{K}$ consists of two parts: the 'macro' component $\widehat{K}_M$ and the 'micro' component $\widehat{K}_m$ and may be written as

$$\widehat{K} = \widehat{K}_M + \widehat{K}_m = \frac{1}{2}\iiint_V \rho\partial_t u_i\partial_t u_i dV + \frac{1}{6}\iiint_V \rho h^2\partial_t u_{i,j}\partial_t u_{i,j}dV, \tag{9}$$

where h is half the edge length of the characteristic cell associated with the micro-structure (Mindlin, 1964). This constant controls the micro-inertia effects of the gradient elastic solid.

The variation $\delta\widehat{U}$ of the strain energy is (after an integration by parts in order to eliminate the derivatives of δu_3)

$$\begin{aligned}\delta\widehat{U} = &\int_0^L\Big[\Big(E\big(I_{11} + g^2A\big)u_{3,11}\Big)_{,11} - (EI_{11}u_{3,111})_{,111}\\ &+ k_w u_3\Big]\delta u_3 dx_1 + \Big[g^2EI_{11}u_{3,111}(\delta u_3)_{,11}\Big]_0^L\\ &+ \Big[\Big\{E\big(I_{11} + g^2A\big)u_{3,11} - \big(g^2EI_{11}u_{3,111}\big)_{,1}\Big\}(\delta u_3)_{,1}\Big]_0^L\\ &+ \Big[\Big\{-\Big(E\big(I_{11} + g^2A\big)u_{3,11}\Big)_{,1} + \big(g^2EI_{11}u_{3,111}\big)_{,11}\Big\}\delta u_3\Big]_0^L.\end{aligned} \tag{10}$$

The variation of the kinetic energy, when defined from time $t = t_0$ to $t = t_1$ is, upon integrating by parts with respect to time and