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Closed-form frequency solutions for simplified strain gradient beams with higher-order inertia

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ABSTRACT

This paper develops an Euler–Bernoulli beam model within the context of a simplified strain gradient theory with higher-order inertia. In contrast to the classical beam models, the proposed gradient beam models can capture the size effects by introducing not only the internal length l_2 related to strain gradient but also the internal length l_1 related to velocity gradient. The governing equation of motion and boundary conditions are derived by using the variational principles. The closed-form solutions for free vibrations of beams with three typical boundary conditions are obtained. Numerical results show that the choices of the higher-order boundary conditions have a minor effect on the natural frequencies of beams. In addition, the inclusion of the strain gradient parameter l_2 increases the effective stiffness of beams and hence it increases the natural frequencies of beams; whereas the inclusion of the velocity gradient parameter l_1 acts as an equivalent compression force in the governing equation and therefore it leads to the decrease of the natural frequencies of beams. Moreover, the significant Poisson effect on the natural frequencies of beams is observed when the thickness of the beam is comparable to the strain gradient parameter l_2. The closed-form solutions for natural frequencies of beams presented in this work may serve as benchmark results for other numerical methods.

1. Introduction

The classical theory of elasticity fails to capture the size effects of materials when their characteristic sizes scale down to the order of microns, as evidenced by the experiment works of Fleck et al. (1994), Stolken and Evans (1998), Lam et al. (2003), McFarland et al. (2005) and Sun et al. (2008), to name a few. Owing to the wide applications of these materials in the micro- and nano-electro-mechanical systems, it is crucial to accurately predict their static and dynamic responses by using the continuum mechanics.

The size-dependent continuum theories, which have been widely used in the analysis of engineering structures such as beams, plates and shells, are believed to be useful and effective tools in the study of the static and dynamic responses of engineering structures. However, the success of the size-dependent continuum theories depends on, to some extent, the types of the materials behave. For instance, the strain gradient theories developed by Kröner (1963) and Mindlin (1964) were found useful to model materials exhibiting stiffening phenomenon; whereas the nonlocal elasticity originated by Eringen (1983) was found effective to the analysis of softening materials. The remarkable feature of these theories which differ from the classical theories is that additional gradient parameter(s) is/are involved in the constitutive equations in order to model periodic structures (e.g., crystal lattices, molecules of a polymer, grains of a granular material and van der Waals interaction between the adjacent atoms). Among these size-dependent theories, the gradient elasticity is a promising candidate for modeling the stiffening materials. The advantages of this theory lie in that it is capable of eliminating the unphysical singularity at the dislocation core (Gutkin and Aifantis, 1996; Lazar and Maugin, 2005), predicting the bounded displacement (as opposed to the classical continuum theory) at the point of application of the load (Georgiadis and Vardoulakis, 1998), and predicting the existence of torsional surface waves in a homogeneous half-space (Georgiadis et al., 2000), etc. Within the framework of the gradient elasticity, the classical equations are generalized by introducing additional spatial derivatives of strains and velocities.

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Here, it was emphasized that the higher-order inertia term (represented as velocity gradient in this paper) should be included in the constitutive equations, because the previous works with gradient models of torsional waves (Georgiadis et al., 2000) and the flexible wave dispersions (Askes and Aifantis, 2009) showed that this term was indispensable for waves propagating at high frequencies. Indeed, including it in the wave problems gave results in consistent with those of atomic-lattice models (Askes and Aifantis, 2009; Georgiadis et al., 2004). Therefore, the high-order inertia term is considered in the present work by assuming that the kinetic energy features the velocity and the velocity gradient.

Since Mindlin's linear generalized gradient theory involves a larger number of material constants, determination of all these material constants is challenging in the practical applications. As a result, requirement of the simplified versions of his generalized gradient theory has attracted numerous attentions. By using the variational principles, Lam et al. (2003) developed a modified strain gradient theory in which only three additional material constants (higher-order material length scale parameters) were involved for isotropic linear elastic materials. Then, they applied their beam model to the bending problems of cantilevers with applied moment and concentrated force at the free end, respectively. In addition, effects of higher-order boundary conditions on the beam bending deflection, moment and shear force were demonstrated in detail. Later, as a simplified version of the so-called Form-II formulation by Mindlin and Eshel (1968), Polizzotto (2012) addressed second-grade elastic materials featured by a potential energy depending upon the strain and the strain gradient, and a kinetic energy depending upon the velocity and the velocity gradient. Until now, there have been an increasing number of works based on this gradient theory, and here we only take the literatures concerning with the dynamic behaviors of beams (Ansari et al., 2013, 2012; Asghari et al., 2012; Fakhrabadi et al., 2013; Kahrobaiyan et al., 2015, 2011; Liang et al., 2014; Wu et al., 2013), plates (Ashoori Movassagh and Mahmoodi, 2013; Ieşan, 2014; Papargyri-Beskou et al., 2010; Sahmani and Ansari, 2013; Wang et al., 2011; Xu et al., 2014; Zhang et al., 2015a) and shells (Daneshmand et al., 2013; Ghavanloo and Fazelzadeh, 2013; Papargyri-Beskou et al., 2012; Xu and Deng, 2015; Zeighampour and Tadi Beni, 2014; Zhang et al., 2015b) during recent years as examples. In these works, simply supported beams, plates and shells are considered, and the analytical expressions can be readily obtained. For the structures with other boundary conditions, however, few analytical results are available for bending, buckling and free vibrations of such structures. Importantly, such analytical solutions are helpful for a better understanding of the size effects phenomena observed both in experimental works and atomic simulations (Askes and Aifantis, 2009; Lam et al., 2003). As a result, the boundary value problems of beam structures in bending (Akgöz and Civalek, 2012; Giannakopoulos and Stamoulis, 2007; Kong et al., 2009; Lazopoulos and Lazopoulos, 2010; Li et al., 2011; Liang et al., 2014; Papargyri-Beskou et al., 2003b), buckling (Lazopoulos and Lazopoulos, 2010; Li et al., 2011; Papargyri-Beskou et al., 2003b) and free vibrations (Artan and Batra, 2012; Kong et al., 2009; Lazopoulos, 2012; Li et al., 2011; Papargyri-Beskou et al., 2003a) using the strain gradient theories have been the growing topic of current interest. Although these works provide useful information in dealing with static (i.e., bending and buckling) problems, the complete boundary value problems for dynamic behaviors of beams with various boundary conditions have not been, to the authors' knowledge, well documented, especially for the dynamic analysis of the differences of the two alternative higher-order boundary conditions selected. By using the strain gradient elasticity, Kong et al. (2009) and Liang et al. (2014) developed the Euler–Bernoulli beam models without considering the higher-order inertial effect. For boundary value problems of beams using hybrid nonlocal beam models, one can refer to Zhang et al. (2010), in which only one higher-order boundary condition was analyzed. And also, effect of the higher-order inertia on the dynamic behaviors of beams had been studied by Lazopoulos (2012). However, the differential order of the spatial coordinate of the inertia in Lazopoulos (2012) is different from the present work; this difference results in the different boundary value problems. It is accepted that the governing equation of motion of Euler–Bernoulli beam models in the context of the second gradient strain theory is of the sixth order instead of the fourth. Consequently, this increasing order of the (partial or ordinary) differential equation leads to mathematical difficulties in solving the boundary value problems in the strain gradient models, even in the simplified models. In general, the complementing higher-order boundary conditions should be involved in the boundary value problems of the sixth order of the (partial or ordinary) differential equation.

The present paper develops an Euler–Bernoulli beam model based on the simplified strain gradient theory presented by Polizzotto (2012), in which the effects of both strain gradient and higher-order inertia are considered. The governing equation of motion and all the boundary conditions of the gradient Euler–Bernoulli beams are derived by using the variational principles. The boundary value problems for free vibrations of beams with three typical boundary conditions are addressed to assess the influences of the alternative higher-order boundary conditions and the two additional gradient parameters on the dynamic behaviors of beams. In addition, comparisons of the results reported in the published works with the present reduced beam models show the validity of the present closed-form solutions for free vibrations of gradient Euler–Bernoulli beams. Moreover, numerical results may serve as benchmark values for further numerical studies.

2. Simplified strain gradient elastic theory

In this section, we present the basic equations which describe the dynamics of the simplified strain gradient elastic theory according to Polizzotto (2012). The strain energy U of a linear three-dimensional continuum occupying a volume V bounded by the surface Γ is given by

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + \tau_{ijk}\eta_{ijk}\right)\mathrm{d}V, \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2}$$

$$\eta_{ijk} = \varepsilon_{ij,k} = \frac{1}{2}\left(u_{i,jk} + u_{j,ik}\right), \tag{3}$$

are the strain tensor and strain gradient tensor, respectively. u_i is the displacement vector, and the Latin indices range from 1 to 3.

In addition, the constitutive equations for the lower-order stress tensor σ_{ij} and the higher-order stress tensor τ_{ijk} are given by

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{4}$$

$$\tau_{ijk} = l_2^2\sigma_{ij,k} = l_2^2\left(\lambda\delta_{ij}\varepsilon_{ll,k} + 2\mu\varepsilon_{ij,k}\right). \tag{5}$$

where l_2 is the internal characteristic strain gradient parameter. λ and μ are the Lame' parameters related to Young's modulus E and Poisson's ratio ν by