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The refraction phenomenon of singularities in thin elastic shells with developable mid-surface in presence of rigid folds: Case of parabolic shells

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ABSTRACT

This paper deals with the refraction phenomenon that appears in the theory of thin shells of which the mid-surface is developable in presence of rigid folds when the thickness ε tends to zero. Roughly speaking we talk about thin parabolic shells. On each side of the fold, the nature of the mid-surface of the shell is developable and boundary conditions ensure the geometric rigidity. The limit problem ($\varepsilon = 0$) which is also parabolic has some peculiarities that induce singularities. These singularities propagate along the asymptotic curves also called characteristics. When these singularities encounter a fold on which transmission conditions are given, they pass through the fold and they continue to propagate along the characteristic curves of the opposite part of the shell corresponding to the adjacent parts of the fold. A theoretical approach is proposed in order to study the refraction phenomenon in thin parabolic shells. Numerical simulations have been constructed to illustrate and observe the refraction phenomenon. The main result is that, once the singularities have crossed the fold, they continue to propagate along the characteristic curves of the other side of the fold without any loss or gain of regularity but with a variation in amplitude.

1. Introduction

It is known that the behaviour of shells for very small values of the relative thickness ε is highly bonded to the geometrical properties of the mid-surface and the loadings. Some works were addressed in this direction see for instance (Karamian, 1998; Sanchez-Palencia, 2000; Karamian et al., 2000, 2003, 2002a, 2002b; Bechet et al., 2008; Bechet et al., 2009). To our knowledge there are no studies concerning the behaviour of these singularities in presence of folds. This paper is devoted to the study and comprehension of refraction phenomenon which appears in developable mid-surface, for a given singularity (for instance loading with special shape and profile). It is known that these singularities propagate along characteristic curves. The issue is what happens once they encounter the fold on which transmission conditions are prescribed. Do they keep propagating after crossing the fold on the adjacent side of the shell along their characteristic curves? Is there any change in their shapes and amplitudes? We must keep in mind that the structure of the system is essentially parabolic,[1] but presents some peculiarities which induce singularities four orders stronger (Bechet et al., 2008, 2009). For instance, discontinuities of the first kind (i.e. Heaviside singularities) of the normal loading imply δ'''-like singularities of the normal component of the displacement. The motivation of the study consists in explaining how these singularities cross the folds on which transmission conditions are given and how they behave when they have crossed them. Indeed, sometimes their amplitude can present some variations whereas they keep their degree of singularities. We are mainly concerned with a developable mid-surface $\mathscr{S}$. So let us recall that a ruled surface is generated by the displacement of a straight line so-called generator. If we denote by $\vec{e}(y^2)$ the unit vector along the generator containing a given point $M(y^2)$, where

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[1] We recall that a shell is parabolic when one of its principal curvature vanishes. It is usually the case of a cylinder or any developable surface.

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$M(y^2)$ is a point on a curve $\mathscr{C}$ defined by $\overrightarrow{OM} = \overrightarrow{\rho}(y^2)$, then a point $P \in \mathscr{S}$ belonging to the generator issued from M is given by $\overrightarrow{OP} = \overrightarrow{\rho}(y^2) + y^1 \overrightarrow{e}(y^2)$ which is the equation of the surface $\mathscr{S}$. A developable surface is a particular case of ruled surface: a ruled surface is said developable when its tangent plane is the same along each generator. A necessary and sufficient condition for a ruled surface to be developable is that $\left(\overrightarrow{\rho}, \overrightarrow{e}, \frac{d\overrightarrow{e}}{dy^2}\right) = 0$. Let us also recall some elements of general surface theory. For each point $P \in \mathscr{S}$ there are two directions, also called asymptotic directions which are tangent to $\mathscr{S}$, where the normal curvature vanishes. Point P is said to be elliptic, hyperbolic or parabolic when their asymptotic directions are imaginary, real and distinct and real coincident respectively. The two families of curves which are tangent at each point to the asymptotic directions are called asymptotic curves. It is classical that a surface is developable when all its points are parabolic. It is also well known that the generators of a ruled surface are asymptotic curves. As a consequence, all along this paper the asymptotic curves are double and coincide with the generators of the surface $\mathscr{S}$. Moreover, they coincide with the double characteristics of the membrane system.

The context of this paper that will be explained in Section 2 is the following. We consider problems of two thin elastic shells for which the mid-surface is developable, we then talk about parabolic shells (i.e. the Gauss curvature is equal to zero whereas the principal curvatures are not equal to zero everywhere), joined together along a curve Γ see Fig. 1 which is identified as a fold in the sequel. The limit problem corresponding to $\varepsilon = 0$, also called membrane problem may be written for both shells as follows:

$$\begin{cases} D_1 T^{11\pm} + D_2 T^{12\pm} + f^{1\pm} = 0, \\ D_1 T^{21\pm} + D_2 T^{22\pm} + f^{2\pm} = 0, \\ b^{\pm}_{22} T^{22\pm} + f^3 = 0, \end{cases} \tag{1}$$

$$\begin{cases} D_1 u^{\pm}_1 = C^{\pm}_{11\lambda\mu} T^{\lambda\mu\pm}, \\ D_2 u^{\pm}_2 - b^{\pm}_{22} u^{\pm}_3 = C^{\pm}_{22\lambda\mu} T^{\lambda\mu\pm}, \\ D_1 u^{\pm}_2 + D_2 u^{\pm}_1 = 2C^{\pm}_{12\lambda\mu} T^{\lambda\mu\pm}, \end{cases} \tag{2}$$

in a domain $\Omega = \Omega^- \cup \Omega^+$ of the (ξ^1,ξ^2)-plane coordinates. The nature of systems (1), (2) of partial differential equations is the same as the mid-surface $\mathscr{S} = \mathscr{S}^- \cup \mathscr{S}^+$ of the shell. That means, the system is parabolic, hyperbolic or elliptic when $\mathscr{S}$ is itself parabolic, hyperbolic or elliptic. Moreover, the characteristic lines of the systems (1), (2) are those of $\mathscr{S}$ so, in the case of cylinder or developable surface studied, the limit problem is parabolic and the characteristic lines are the generators of $\mathscr{S}$. The unknowns are the symmetric membrane stresses $T^{\lambda\mu\pm}$ (λ, $\mu \in [1,2]$) and the displacements u_i ($i \in [1\ldots3]$). Let us recall that the symbols D_1 & D_2 denote the covariant derivative with respect to the variables (ξ^1,ξ^2). The coefficients $C^{\pm}_{\alpha\beta\lambda\mu}$ are the compliance ones for each part of the shell and are given smooth functions. The coefficient $b^{\pm}_{22}$ corresponds to the second fundamental form of each part of the shell. These functions are smooth functions and different from zero.

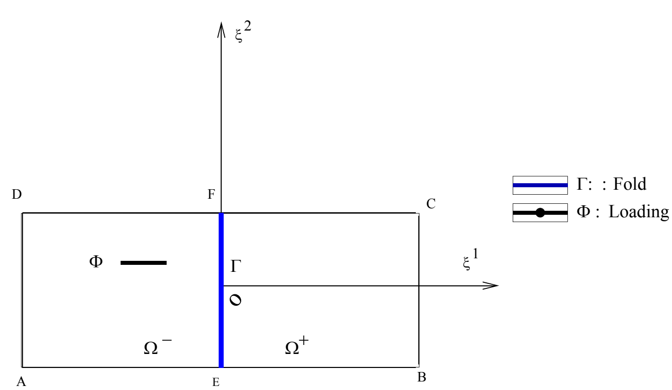

Fig. 1. Reference domain for the parabolic case.

Now, let us define, upon the boundaries ∂S^- and ∂S^+, two local direct orthonormal reference systems $(\overrightarrow{n}^-, \overrightarrow{t}^-, \overrightarrow{a}^-_3)$ and $(\overrightarrow{n}^+, \overrightarrow{t}^+, \overrightarrow{a}^+_3)$, where $\overrightarrow{n}^{\pm}$ and $\overrightarrow{t}^{\pm} = \overrightarrow{a_3}^{\pm} \wedge \overrightarrow{n}^{\pm}$ are the intrinsic vectors corresponding to the outward unit normal vectors, the unit tangent vectors and $\overrightarrow{a}^{\pm}_3$ respectively. Let us introduce $\theta(\overrightarrow{n}^-, \overrightarrow{n}^+)_{\overrightarrow{t}^-}$ with respect to the reference system $(\overrightarrow{n}^-, \overrightarrow{t}^-, \overrightarrow{a}^-_3)$. The parameter θ represents the angle between shells. This angle is not necessarily constant. In the sequel, as we are interested by the study of rigid folds we will consider it as constant. This remark leads us to postulate the transmission conditions which are:

$$\overrightarrow{u}^-_\Gamma = \overrightarrow{u}^+_\Gamma \quad \text{and } \theta\left(\overrightarrow{n}^-, \overrightarrow{n}^+\right)_{\overrightarrow{t}^-} = Const. \quad \text{on } \Gamma$$

These relations ensure the continuity of the displacements and of the tangential rotation along the fold for all points of Γ. The fold has a rigid behaviour, in the sequel, we will talk about rigid fold or merely fold. Let us recall that the loadings $\overrightarrow{f}^{\pm} = (f^1, f^2, f^3)^{\pm}$ are datum such that in general do not belong to the energy space of the membrane problem. The system (1)–(2) has six equations and six unknowns. Nevertheless, $T^{22\pm}$ is given by (1_3) and $u^{\pm}_3$ only appears in (2_2) which can be taken as a definition of u_3. Then the unknowns are essentially $T^{11\pm}$ and $T^{12\pm}$, $u^{\pm}_1$ and $u^{\pm}_2$. The first two equations of (1) only involve $T^{11\pm}$ and $T^{12\pm}$ since $T^{22\pm}$ now is a data and constitute a first-order parabolic system for them with a double characteristic $\xi^2 = const$. We shall notice that, in a general case, the boundary conditions do not allow us to fully determine $T^{11\pm}$ and $T^{12\pm}$, nevertheless we assume that the right-hand side of (2) is known and the first and third equations of (2) form again a first-order parabolic system $u^{\pm}_1$ and $u^{\pm}_2$ with the similar double characteristics. As usual, we focus our attention on forces, which correspond to the normal component of the loading and the normal displacement of $u^{\pm}_3$. At this stage, we can easily understand that on the adjacent side of the fold corresponding to the opposite side where the singularities are generated on the shell, the transmission conditions on the displacement are going to generate a kind of singularity that they continue to propagate on the characteristic lines. The paper is outlined as follows: The real description of the refraction of singularities is explained in section 4. Precision of mechanical problems and the specific data are given in section 2. In section 3 we give the mathematical framework in order to study the expansions of singularities. In section 5 in view of numerical simulations we have given the adapted numerical variational formulation. Section 6 is dedicated to a study of a practical example to illustrate the theoretical results. To finish, in section 7, some numerical simulations illustrate such kind of refraction phenomenon.

2. Mechanical problem

Let us recall here, some elements of shell theory which are necessary for understanding the sequel of the paper. We limit our study to linear elastic isotropic shells described by the linear Koiter shell model. More description for interested readers can be found in shell treatises (Bernadou, 1994; Koiter, 1970; Ciarlet, 2000; Goldenveizer, 1962; SANSanchez-Hubert and Sanchez, 1997).

A shell is a continuum medium which can be geometrically defined by a mid-surface $\mathscr{S}$ in the physical space $\mathscr{E}$ and a small parameter ε representing the thickness of the shell around this