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An axisymmetric stress analysis in a single fibre composite of finite length under a thermal expansion mismatch

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ABSTRACT

In this paper, an exact solution is derived for the characterization of thermal stresses in a single-fibre composite of finite length. All the required boundary and interfacial conditions of the thermo-elastic problem are thus satisfied exactly. The proposed method involves a particular solution that is added to a three-dimensional (3D) complementary displacement field which satisfies automatically the Navier's equations. Based on experimental data provided by fibre Bragg grating sensors, an axisymmetric analysis is used then to determine the residual stress field inside the composite due to matrix shrinkage. The numerical results clearly indicate that all stress components vary significantly near the ends. An abrupt change of the shear stresses is thus predicted close to the edges. The results of the model are also found to be in good agreement with those obtained from finite element simulations. A comparison of the proposed approach with three other published theoretical models is also presented.

1. Introduction

Fibre-reinforced thermosets (e.g. epoxies) are commonly used in advanced engineering applications, owing to their resistance to corrosive environment, versatility and light weight. They can be advantageously tailored to meet the application requirements and have shown superior performance over traditional materials in certain situations (e.g. the conveyance of aggressive media). However, it is well recognized that significant residual stresses may arise during the fabrication process, influencing the in-situ mechanical response of the component. Broadly speaking, residual stresses are attributed to differences in the coefficients of thermal expansion (CTEs) of the fibres and matrix as well as resin differential cure shrinkage. Depending on their nature, these stresses can enhance interfacial contact between the components but also initiate matrix or interface cracking and cause premature failure. In other words, they may have a pronounced effect on the micro-stress field within a composite structure and must be added to the stresses induced by the external mechanical loads.

A number of two-dimensional (2D) analytical solutions are presently available in the literature to describe the local thermal stress state in fibre-reinforced composites (Mikata and Taya, 1985). Such thermo-elastic models often deal with two or several concentric long circular cylinders representing different material phases (e.g. a coating layer). More or less complex interfacial conditions may also be introduced to take into account imperfections in the adhesion between fibre and matrix. For example, an imperfect interphase is simulated by introducing a hollow cylindrical layer with radius-dependent thermo-elastic properties (Jayaraman and Reifsnider, 1993). However, most of these models assume generalized plane strain conditions, only valid for infinitely long cylinders. These solutions cannot satisfy properly the boundary conditions at the ends of finite length cylinders. According to Saint Venant's principle, they are convenient approximations of the problem far from the ends where higher damaging stresses are generally expected. In particular, the sharp changes of the interfacial shear stress near the ends cannot be captured by these simplified analyzes.

It is rather difficult to derive three-dimensional (3D) closed form solutions for similar problems, because of the mathematical difficulties involved. Recognizing the significance of stress concentration at fibre ends, some investigators have adopted numerical methods (FE analysis) to evaluate properly the effects of end conditions as well as interface properties (Kovalev et al., 1998). A theoretical model has been developed by Quek (2004) for the analysis of thermal stresses in a single fibre-matrix composite of finite length. Elastic solutions have also been obtained for axisymmetric boundary value problems involving stress transfer in two-phase cylindrical composites, under specific prescribed forces and/or displacements at their external surfaces (Nairn and Liu, 1997; Wu et al., 2000). Kurtz and Pagano (1991) have treated essentially the same problem using stress function formalism. Solutions are typically found by using appropriate eigenfunction expansions that are superimposed to appropriate particular solutions to meet the required boundary conditions.

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This work presents an exact solution for the stress field in a single-fibre composite cylinder of finite length, subjected to thermal loading. The solution is then used to predict the residual stresses due to matrix shrinkage. The proposed approach combines a particular solution for the thermal effect with a more general solution that was originally constructed by Folias (1975) to solve 3D problems of elasticity. The method involves a double series of eigenfunctions with complex unknown coefficients to be determined from the boundary conditions.

The paper is organized as follows. Section 2 outlines the composite model and the associated thermo-elastic governing equations. The general method of solution is presented in Section 3 for the problem under consideration, with some details provided in Appendix A. The unknown coefficients appearing in the solution are determined uniquely by solving a set of infinite linear equations. Results and discussion are presented in Section 4: First the predictions of the model are shown using experimental data obtained for a single fibre composite using a long Fibre Bragg grating (Colpo et al., 2007). Next the model is compared with finite elements (FE) simulations. The predictions of the model are also discussed and compared with three other available benchmark solutions. Finally, summary and conclusions are given in Section 5.

2. Model and basic equations

The model composite considered in this study is made up of a solid cylinder (fibre) of radius a and length $2h$ and an equally long hollow cylinder (matrix) of inner radius a and outer radius b. Introducing cylindrical coordinates r, θ, z (see Fig. 1), the composite structure is bounded by the planes $z = |h|$, occupying the space $r \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$, $|z| \leqslant h$. The analysis is restricted to small strains and both fibre and matrix materials are assumed to be homogeneous, isotropic and linear elastic. Intact interface is assumed between the fibre and matrix. This assumption is based on experimental evidence on a single fibre composite specimen where the reinforcing glass fibre contains a long Bragg grating which gives the strain distribution due to shrinkage along the grating (Colpo et al., 2007). Such strains do not change after postcuring over long periods of time showing an intact interface. Moreover, an axisymmetric mode of deformation is supposed to exist so that the only non-vanishing displacement and stress components are u_r, u_z and σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, τ_{rz}, respectively. All these quantities are independent of the coordinate θ.

The structure is subjected to a temperature rise while all points of the ends are free of stress and constraint. Additionally, the outer cylindrical boundary $r = b$, $|z| = h$ is assumed to be free of stress. In the subsequent, matrix and fibre sub-domains are denoted by superscripts (1) and (2), respectively.

In the absence of body forces, the thermo-elastic axisymmetric problem reduces to a solution of the following coupled differential equations governing the axial and radial displacements $u_z^{(i)}$ and $u_r^{(i)}$ $(i = 1, 2)$

$$\frac{m_i}{m_i-2}\frac{\partial}{\partial r}e^{(i)} + \nabla^2 u_r^{(i)} - \frac{u_r^{(i)}}{r^2} = 2\frac{m_i+1}{m_i-2}\frac{\partial}{\partial r}(\alpha_i T)$$
$$\frac{m_i}{m_i-2}\frac{\partial}{\partial z}e^{(i)} + \nabla^2 u_z^{(i)} = 2\frac{m_i+1}{m_i-2}\frac{\partial}{\partial z}(\alpha_i T) \qquad (1)$$

where T is the temperature rise, $m_i = 1/\nu_i$ (Poisson's number) the reciprocal of the Poisson's ratio ν_i and α_i the coefficient of thermal expansion for each of the material domains. In (1) ∇^2 is the Laplacian operator and $e^{(i)}$ the volumetric dilation (first invariant of the small-strain tensor) expressed by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \qquad e^{(i)} = \frac{\partial u_r^{(i)}}{\partial r} + \frac{u_r^{(i)}}{r} + \frac{\partial u_z^{(i)}}{\partial z} \qquad (2)$$

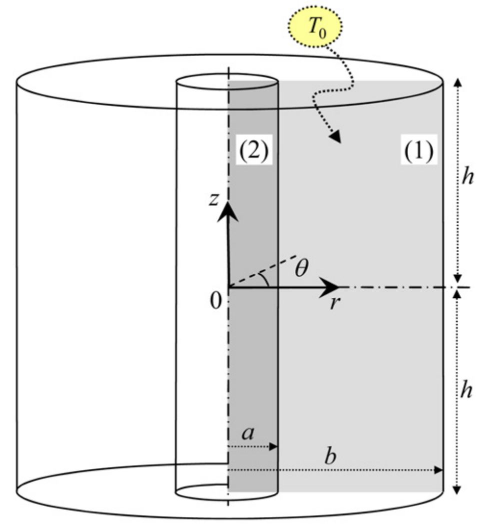

Fig. 1. Definition of the model composite with (1) matrix and (2) fibre domains subjected to a uniform temperature rise T_0.

In the subsequent, a uniform temperature rise T_0 is considered. Consequently, system (1) reduces to the homogeneous Navier's equations of the elasticity theory. The thermo-elastic stress–strain relations are expressed by

$$\frac{1}{2\mu_i}\sigma_{rr}^{(i)} = \frac{1}{m_i-2}e^{(i)} + \varepsilon_{rr}^{(i)} - \frac{m_i+1}{m_i-2}\alpha_i T_0$$
$$\frac{1}{2\mu_i}\sigma_{zz}^{(i)} = \frac{1}{m_i-2}e^{(i)} + \varepsilon_{zz}^{(i)} - \frac{m_i+1}{m_i-2}\alpha_i T_0$$
$$\frac{1}{2\mu_i}\sigma_{\theta\theta}^{(i)} = \frac{1}{m_i-2}e^{(i)} + \varepsilon_{\theta\theta}^{(i)} - \frac{m_i+1}{m_i-2}\alpha_i T_0$$
$$\frac{1}{2\mu_i}\tau_{rz}^{(i)} = \varepsilon_{rz}^{(i)} \qquad (3)$$

where μ_i is the shear modulus for each material domain and the infinitesimal strain components are given by

$$\varepsilon_{rr}^{(i)} = \frac{\partial u_r^{(i)}}{\partial r}, \qquad \varepsilon_{zz}^{(i)} = \frac{\partial u_z^{(i)}}{\partial z}$$
$$\varepsilon_{\theta\theta}^{(i)} = \frac{u_r^{(i)}}{r}, \qquad 2\varepsilon_{rz}^{(i)} = \frac{\partial u_z^{(i)}}{\partial r} + \frac{\partial u_r^{(i)}}{\partial z} \qquad (4)$$

Regarding the boundary conditions, it is required

- four continuity relations at the fibre-matrix interface (perfect bonding)

$$u_r^{(1)}\big|_{r=a} = u_r^{(2)}\big|_{r=a} \qquad \text{(a)}$$
$$u_z^{(1)}\big|_{r=a} = u_z^{(2)}\big|_{r=a} \qquad \text{(b)}$$
$$\tau_{rz}^{(1)}\big|_{r=a} = \tau_{rz}^{(2)}\big|_{r=a} \qquad \text{(c)}$$
$$\sigma_{rr}^{(1)}\big|_{r=a} = \sigma_{rr}^{(2)}\big|_{r=a} \qquad \text{(d)} \qquad (5)$$

- the outer edges (external cylindrical surface and ends) to be traction free, namely

$$\tau_{rz}^{(1)}\big|_{r=b} = 0 \qquad \text{(a)}$$
$$\sigma_{rr}^{(1)}\big|_{r=b} = 0 \qquad \text{(b)}$$
$$\tau_{rz}^{(1)}\big|_{z=\pm h} = \tau_{rz}^{(2)}\big|_{z=\pm h} = 0 \qquad \text{(c)}$$
$$\sigma_{zz}^{(1)}\big|_{z=\pm h} = \sigma_{zz}^{(2)}\big|_{z=\pm h} = 0 \qquad \text{(d)} \qquad (6)$$