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Two and three-dimensional boundary element formulations of compressible isotropic, transversely isotropic and orthotropic viscoelastic layers of arbitrary thickness, applied to the rolling resistance of rigid cylinders and spheres

Gérard-Philippe Zéhil*, Henri P. Gavin

Department of Civil and Environmental Engineering, Duke University, 121 Hudson Hall, Box 90287, Research Drive, Durham, NC 27708-0287, United States

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ABSTRACT

New two-dimensional and three-dimensional boundary element formulations of *compressible* viscoelastic layers of arbitrary thickness are presented in this work. The formulations are derived in increasing order of complexity for: (i) compressible isotropic layers, (ii) transversely isotropic layers, and (iii) fully orthotropic layers. It is further shown that existing 2D and 3D models for *incompressible* isotropic layers may be regarded as particular instances of case (i). The proposed formulations are based on Fourier series and support any linear viscoelastic material model characterized by general frequency-domain master-curves. These approaches result in a compliance matrix for the layer's upper boundary, which includes the effects of steady-state motion. This characterization may be used as a component in various problem settings to generate sequences of high fidelity solutions for varying parameters. The proposed modeling techniques are applied, in combination with appropriate contact solvers, to the rolling resistance of rigid cylinders and spheres on compressible isotropic, transversely isotropic and orthotropic layers. The latter case reveals that the dissipated power varies with the direction of motion, which suggests new ways of optimizing the level of damping in various engineering applications of very high impact. Interesting lateral viscoelastic effects resulting from material asymmetry are unveiled. These phenomena could be harnessed to achieve smooth and 'invisible' guides across three-dimensional viscoelastic surfaces, and hence suggest new ways of controlling trajectories, with a broad range of potential applications.

1. Motivations and background

A three-dimensional boundary element formulation of an *incompressible* viscoelastic layer of arbitrary thickness was proposed by Zéhil and Gavin (2013b). This formulation was applied, in combination with appropriate rolling/sliding contact algorithms (Zéhil and Gavin, 2013a), to determine the resistance incurred by a rigid sphere rolling, in steady-state, on such a layer.

The authors' interest in rolling resistance initially stems from the exploration of new damping principles suitable for the seismic isolation of critical facilities. A recent study conducted by Harvey et al. (2013) on a Ball-N-Cone™rolling isolation bearing (WorkSafe Technologies, 2012) addresses the benefits of damping in rolling isolation systems. In order to limit the peak acceleration levels to which sensitive equipment may be subjected to during the course of an earthquake, higher levels of 'soft' damping can be achieved by increasing the resistance to rolling or sliding of their seismic isolation platforms. In practice, this damping principle can be implemented by inserting a dampening material between contacting components of the isolation system, in relative motion with respect to each other. For instance, in the case of rolling isolation bearings, viscoelastic rubber sheets can be inserted between the rigid roller (e.g. a sphere) and the hard surfaces on which the rolling occurs (e.g. 'dished' or bowl-shaped steel plates).

An early and approximate closed-form expression for the rolling resistance R_r incurred by a rigid sphere rolling on a *compressible* viscoelastic half-space was derived by Greenwood and Tabor (1958) who integrated, under the small strain assumption, the horizontal projection of the stationary normal stress distribution, as given by Hertz (1881), over the front half of the contact 'disk', and evaluated its work per unit distance of rolling. The proposed expression for rolling resistance is in good agreement with experimental results

* Corresponding author.
E-mail address: gerard.zehil@duke.edu (G.-P. Zéhil).

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presented by the authors for spheres moving slower than at 1 cm/s under mean contact pressures below 2.76 MPa (i.e. 400 lb in^{-2}). This expression may be re-written as follows

$$R_r \approx \alpha_{GT} \left(\frac{3^4}{4^7}\right)^{\frac{1}{3}} \left(\frac{1-\nu^2}{E}\right)^{\frac{1}{3}} \frac{P^{\frac{4}{3}}}{R^{\frac{2}{3}}}, \tag{1}$$

where P is the vertical load supported by the rolling sphere, and R stands for its radius. Young's modulus E and Poisson's ratio ν of the layer's material, as well as the loss fraction α_{GT} of the input deformation energy are taken as constants. Flom and Bueche (1959) proposed an alternative simplified theory accounting for the influence of rolling speed and resulting in expressions that otherwise confirm, for intermediate values of the dynamic loss factor (known as $tan\delta$), the dependencies of rolling resistance predicted by Equation (1) on the vertical load P, the radius of the sphere R and the layer's stiffness. These expressions may be written in the generic form

$$R_r \approx \alpha_{FB} \left(\frac{1-\nu^2}{E}\right)^{\frac{1}{3}} \frac{P^{\frac{4}{3}}}{R^{\frac{2}{3}}}, \tag{2}$$

where, as noted by Lakes (2009), α_{FB} depends on $tan\delta$ and therefore on the material parameters of the layer and on the frequency of rolling. Based on the simplifying assumption that the dynamic contact region has a similar size to that given by the static solution of Hertz, Lakes also noted that an upper bound for the viscoelastic rolling resistance of a rigid sphere is given by

$$R_r \approx \left(\frac{3}{4}\right)^{\frac{1}{3}} \left(\frac{1-\nu^2}{E}\right)^{\frac{1}{3}} \frac{P^{\frac{4}{3}}}{R^{\frac{2}{3}}}, \tag{3}$$

where E is interpreted as a dynamic modulus at a circular frequency ω, proportional to V_s/R. Expressions (1)–(3) are furthermore consistent in predicting that rolling resistance decreases with Young's modulus and that it is maximized by a Poisson ratio of zero. Hence, based on this simple and qualitative reasoning, it may be expected that relatively soft and compressible layers, with a Poisson ratio that it close to zero, would yield higher levels of resistance and damping than harder layers or layers made of incompressible materials ($\nu \approx 0.5$) such as rubbers.

The boundary element formulation presented in Zéhil and Gavin (2013b) applies to *incompressible* and *isotropic* layers, which is practically the case of most rubber-like materials. Compressible materials are however characterized by one additional frequency-dependent complex parameter, i.e. the complex Poisson ratio $\nu^*(\omega)$, and can not be modeled with this formulation. Extending the boundary element formulation to *compressible* layers is therefore needed. In Section 4, a compressible isotropic formulation is derived, in three dimensions, to answer this first need. This derivation is somewhat akin to that proposed by Persson (2001) based on Fourier transforms and applied in a simplified approach to the rolling resistance of hard cylinders and sphere on a viscoelastic layer (Persson, 2010). A two-dimensional formulation in plane strain is deduced in Section 7 to complement the incompressible formulations proposed by Qiu (2006, 2009).

Moving further, one may think of cork as an example of relatively soft material characterized by a Poisson ratio that is close to zero. In fact this material is used as a stopper for wine bottles because it shows very little lateral expansion when it is compressed. However, cork does not behave isotropically. Indeed, its prismatic cells are packed in columns in the radial direction, which constitutes a direction of symmetry of the cellular structure. Cork may therefore be modeled as a transversely isotropic medium (e.g. Rosa and Fortes, 1991). In order to achieve accurate rolling resistance predictions on viscoelastic materials such as cork, with different mechanical characteristics in the out-of-plane direction, the boundary element formulation must further be extended to polar anisotropic layers. This additional need is addressed in Section 5 where a three-dimensional transversely isotropic formulation is derived. This formulation is specialized further to plane strain in Section 8.

On the other hand, a hard-wearing layer cannot be too soft. Hence, there seems to be a tradeoff between high resistance to rolling and durability. Given the need to achieve optimal levels of damping under specific conditions (of seismic hazard for instance) while maintaining suitable service life expectancies, the future use of specially designed layers made of viscoelastic *metamaterials* cannot be excluded and should therefore be prepared. Man-made materials, such as auxetic composites made of rubber-filled re-entrant honeycombs for instance, are often characterized by different mechanical properties in three orthogonal directions. Predicting the resistance of such materials to rolling and sliding would ultimately require extending the boundary element formulation to fully orthotropic layers. This need is fully addressed in Section 6.

2. Common setting

The different cases considered in this work share a common setting, which is illustrated in Fig. 1: a mechanical load is translated at constant speed V_s, in direction x, on a viscoelastic layer of arbitrary thickness H, attached to a rigid backing. The load is periodic in directions x and y, with periods L_x and L_y, respectively. The coordinate system $Ox'y'z'$ is fixed while $Oxyz$ moves with the load.

3. A brief review of 3D linear viscoelasticity

Linear elasticity corresponds to a time-independent behavioral material model characterized by the constitutive equation below, also known as Hook's law, written in indicial notation as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{4}$$

where σ_{ij}, ε_{ij} and C_{ijkl} are components of the second order stress tensor, the small-strain tensor, and the fourth order elasticity tensor, respectively.

Alternatively, linear viscoelasticity is characterized by the dependence of the elasticity tensor on time. The state of stress in a linear viscoelastic material, subjected to a strain history of the form $\varepsilon_{kl}(t) = \overline{\varepsilon}_{kl}\mathscr{H}(t)$, where $\overline{\varepsilon}_{kl}$ are constant strain components and $\mathscr{H}(\cdot)$ designates the Heaviside unit step function, is given by

$$\sigma_{ij}(t) = C_{ijkl}(t)\overline{\varepsilon}_{kl}, \quad t \geq 0. \tag{5}$$

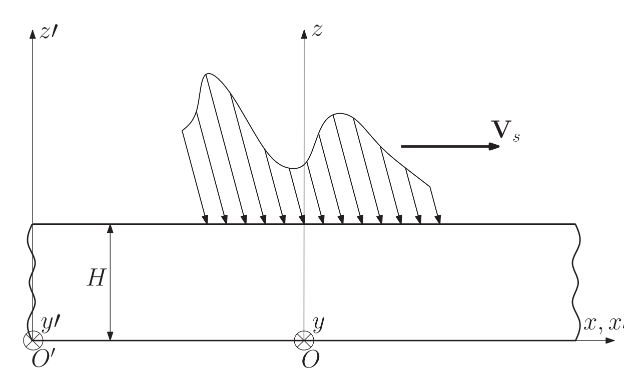

Fig. 1. General model and coordinate systems.