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# Three-dimensional elasticity solution for vibration analysis of functionally graded hollow and solid bodies of revolution. Part II: Application

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ABSTRACT

The semi-analytical method, developed in Part I of this paper, is employed to investigate the free, steady-state and transient vibrations of various FGM bodies of revolution. A comprehensive investigation concerning the convergence, accuracy and efficiency of the method is given for free vibrations of hollow and solid FGM cylinders, cones and spheres with different combinations of free, simply-supported, clamped and elastic-supported boundary conditions. It is shown that the present method enables rapid convergence, stable numerical operation and very high computational accuracy. Both lower- and higher-order frequencies can be accurately obtained by using a small computational effort. The utility and robustness of the method for the application of various polynomial functions are evaluated. New vibration results for FGM bodies of revolution are presented, which could serve as benchmarks for future research. Parametric studies are carried out to highlight the influences of geometrical parameters, boundary conditions, and material profiles on free vibrations of FGM cylinders, cones and spheres. With regard to the forced vibration problems, harmonic responses of hollow and solid cylinders under uniformly axial and normal pressures are calculated, and time domain solutions of FGM cones subjected to several impulsive loads, including a rectangular pulse, a triangular pulse, a half-sine pulse and an exponential pulse, are also examined.



## 1. Introduction

In Part I of the two companion papers, a theoretical method based on the 3-D theory of elasticity was presented for predicting the linear vibration behaviors of hollow and solid FGM bodies of revolution with boundary conditions of arbitrary type. Part II is a natural extension of the preceding paper, and the primary objectives of this work are to: firstly, evaluate the convergence, accuracy and applicability of the theoretical algorithm through a variety of numerical tests concerning the free, steady-state and transient vibrations of FGM bodies of revolution; secondly, present benchmark elasticity solutions for thick hollow and solid FGM bodies of revolution that do not fall into the categories of 2-D shell theories; finally, provide useful guidelines for the practical design of axisymmetric FMG bodies via parametric studies.

For the purpose of subsequent discussion in this paper, some definitions pertaining to the entire analysis should be made. In the numerical examples presented, a simple letter string will be used to describe the type of boundary conditions imposed on the ends of a hollow/solid body of revolution, e.g., the symbol F–S1 for a hollow/solid cylinder identifies the cylinder having free and simply-supported I boundary conditions at $x = 0$ and $x = L$, respectively, and the symbol C–C for a hollow cone indicates the body is clamped at two ends. Similarly, an annular F- E3 FGM sphere implies free and meridionally elastic boundary conditions are respectively imposed on the end faces at $\varphi = \varphi_1$ and $\varphi = \varphi_2$. The effective material properties are evaluated using either the Voigt's rule of mixture or the Mori–Tanaka's homogenization scheme, and the corresponding vibration results derived from the two models for FGM bodies are referred to Voigt and M–T solutions, respectively. Unless stated otherwise, all FGM bodies of revolution considered in the following are fabricated from zirconia (ceramic) and aluminum (metal) with material properties given as: $E_1 = 168$ GPa, $\mu_1 = 0.3$, $\rho_1 = 5700$ kgm$^{-3}$ for zirconia, and $E_2 = 70$ GPa, $\mu_2 = 0.3$, $\rho_2 = 2707$ kgm$^{-3}$ for aluminum.

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## 2. Free vibration analysis of FGM bodies of revolution

### 2.1. Numerical results for hollow and solid FGM cylinders

In this subsection, several numerical examples concerning the free vibrations of hollow and solid FGM cylinders with different physical parameters and boundary conditions have been investigated to demonstrate the convergence, accuracy and versatility of the present method. As a first testing case, FGM cylinders composed of the mixture of zirconia and aluminum with F–C boundary conditions are examined. Before going into the theoretical analysis, it should be noted here that the accuracy of vibration predictions of the present method naturally depends upon both the terms of polynomials taken for the displacement components ($u$ and $w$) and the number of segments decomposed in the length direction of the cylinders. Consequently, an important aspect of the present analysis is related to the solution convergence rate. In Table 1, convergence studies of the natural frequencies $\Omega_{n,m}$ corresponding to circumferential waves $n = 1, 2$ and longitudinal modes $m = 1, 2, 3, 10, 15$ for FGM cylinders are performed to determine the optimal number of polynomial terms and segments required for satisfactory solutions. The geometrical parameters (length $L$, outer radius $R_o$ and inner radius $R_i$) of the cylinders are given along with the table. Note that the thickness-to-radius ratios $H/R_o$ ($H = R_o - R_i$) in Table 1 are 0.01, 0.5 and 1.0 corresponding to thin, very thick cylindrical shells and solid cylinder, respectively. It is assumed that the material properties of the FGM cylinders vary continuously in the radial direction according to the ordinary power-law distributions in terms of volume fractions of the constituents by setting $a = 1$ and $b = 0$ in$\text{FGM}_{\text{I}\,(a/b/c/p)}$ and $\text{FGM}_{\text{II}\,(a/b/c/p)}$. In such cases, a $\text{FGM}_{\text{I}\,(a=1/b=0/c/p=1)}$ cylinder has zirconia on its inner surface and aluminum on its outer surface, whereas the order of the constituent materials is reversed in a $\text{FGM}_{\text{II}\,(a=1/b=0/c/p=1)}$ cylinder. The Mori–Tanaka's scheme is employed for the theoretical computation of the effective material properties. The displacement components (i.e.,$u$,$v$ and $w$) of the FGM cylinders are expanded by Chebyshev orthogonal polynomials of first kind (COPFK), and '$P \times Q \times N$' in the second column of Table 1 identifies the number of COPFK terms and cylinder segments. For example, a typical solution size of $6 \times 6 \times 4$ means six terms of polynomials in

**Table 1**
Convergence of frequencies $\Omega_{n,m}$ (Hz) for hollow and solid FGM cylinders with different thickness-to-radius ratios $H/R_o$ ($L/R_o = 2$,$H = R_o - R_i$,$R_o = 1$ m; M–T solutions; boundary condition: F–C).

| $n$ | $P \times Q \times N$ | $\text{FGM}_{\text{I}\,(a=1/b=0/c/p=1)}$ | | | | | $\text{FGM}_{\text{II}\,(a=1/b=0/c/p=1)}$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $m = 1$ | $m = 2$ | $m = 3$ | $m = 10$ | $m = 15$ | $m = 1$ | $m = 2$ | $m = 3$ | $m = 10$ | $m = 15$ |
| *Thickness-to-radius ratio*: $H/R_o = 0.01$ | | | | | | | | | | | |
| 1 | $4 \times 4 \times 4$ | 236.34 | 606.16 | 737.08 | 1190.60 | 2807.29 | 236.42 | 606.15 | 736.38 | 1189.41 | 2808.15 |
| | $4 \times 4 \times 8$ | 235.71 | 606.12 | 735.31 | 945.91 | 1455.77 | 235.79 | 606.11 | 734.61 | 944.91 | 1455.86 |
| | $6 \times 6 \times 4$ | 235.60 | 606.12 | 735.09 | 926.28 | 1569.49 | 235.67 | 606.10 | 734.38 | 925.28 | 1569.06 |
| | $6 \times 6 \times 8$ | 235.55 | 606.10 | 735.05 | 923.06 | 1241.82 | 235.62 | 606.08 | 734.35 | 922.01 | 1240.88 |
| | $8 \times 8 \times 4$ | 235.55 | 606.10 | 735.06 | 923.01 | 1252.63 | 235.62 | 606.08 | 734.35 | 921.97 | 1251.68 |
| | $8 \times 8 \times 6$ | 235.54 | 606.09 | 735.04 | 922.98 | 1239.14 | 235.62 | 606.08 | 734.35 | 921.94 | 1238.19 |
| | $12 \times 12 \times 8$ | 235.51 | 606.09 | 735.02 | 922.95 | 1239.00 | 235.60 | 606.07 | 734.33 | 921.91 | 1238.05 |
| 2 | $4 \times 4 \times 4$ | 118.48 | 390.30 | 622.99 | 1714.26 | 2930.31 | 118.55 | 390.36 | 622.59 | 1713.23 | 2930.51 |
| | $4 \times 4 \times 8$ | 117.89 | 389.24 | 621.39 | 1012.62 | 1698.64 | 117.96 | 389.31 | 620.99 | 1011.76 | 1699.36 |
| | $6 \times 6 \times 4$ | 117.78 | 389.13 | 621.25 | 987.75 | 1714.42 | 117.85 | 389.20 | 620.86 | 986.83 | 1713.43 |
| | $6 \times 6 \times 8$ | 117.73 | 389.11 | 621.24 | 973.27 | 1366.37 | 117.80 | 389.18 | 620.84 | 972.30 | 1365.43 |
| | $8 \times 8 \times 4$ | 117.73 | 389.12 | 621.24 | 973.31 | 1386.56 | 117.80 | 389.18 | 620.85 | 972.34 | 1385.64 |
| | $8 \times 8 \times 6$ | 117.72 | 389.11 | 621.24 | 973.07 | 1360.93 | 117.79 | 389.18 | 620.84 | 972.11 | 1359.98 |
| | $12 \times 12 \times 8$ | 117.69 | 389.11 | 621.23 | 973.03 | 1360.48 | 117.77 | 389.18 | 620.84 | 972.06 | 1359.53 |
| *Thickness-to-radius ratio*: $H/R_o = 0.5$ | | | | | | | | | | | |
| 1 | $4 \times 4 \times 4$ | 244.07 | 687.76 | 1051.37 | 2777.50 | 3676.43 | 251.49 | 694.38 | 991.75 | 2786.75 | 3737.45 |
| | $4 \times 4 \times 8$ | 243.93 | 687.63 | 1051.06 | 2767.30 | 3652.59 | 251.29 | 694.28 | 991.35 | 2765.83 | 3700.92 |
| | $6 \times 6 \times 4$ | 243.77 | 687.36 | 1050.54 | 2763.82 | 3642.46 | 251.10 | 694.04 | 990.78 | 2758.44 | 3688.64 |
| | $6 \times 6 \times 8$ | 243.73 | 687.32 | 1050.48 | 2763.77 | 3642.31 | 251.05 | 694.02 | 990.69 | 2758.32 | 3688.42 |
| | $8 \times 8 \times 4$ | 243.70 | 687.29 | 1050.41 | 2763.67 | 3642.04 | 251.02 | 693.99 | 990.62 | 2758.04 | 3688.11 |
| | $8 \times 8 \times 6$ | 243.69 | 687.28 | 1050.39 | 2763.66 | 3642.01 | 251.00 | 693.98 | 990.59 | 2758.01 | 3688.07 |
| | $12 \times 12 \times 8$ | 243.66 | 687.25 | 1050.34 | 2763.63 | 3641.91 | 250.97 | 693.96 | 990.53 | 2757.90 | 3687.96 |
| 2 | $4 \times 4 \times 4$ | 552.87 | 859.66 | 1425.39 | 3000.09 | 3827.53 | 502.42 | 826.49 | 1358.04 | 2989.92 | 3868.62 |
| | $4 \times 4 \times 8$ | 552.74 | 859.39 | 1424.95 | 2996.62 | 3809.88 | 502.27 | 826.16 | 1357.58 | 2987.69 | 3847.22 |
| | $6 \times 6 \times 4$ | 551.16 | 857.43 | 1422.56 | 2994.60 | 3804.12 | 500.49 | 824.19 | 1355.92 | 2986.49 | 3843.81 |
| | $6 \times 6 \times 8$ | 551.15 | 857.37 | 1422.48 | 2994.54 | 3804.06 | 500.47 | 824.13 | 1355.83 | 2986.40 | 3843.72 |
| | $8 \times 8 \times 4$ | 551.13 | 857.29 | 1422.36 | 2994.47 | 3803.96 | 500.44 | 824.04 | 1355.74 | 2986.32 | 3843.65 |
| | $8 \times 8 \times 6$ | 551.13 | 857.27 | 1422.33 | 2994.46 | 3803.95 | 500.43 | 824.02 | 1355.71 | 2986.29 | 3843.63 |
| | $12 \times 12 \times 8$ | 551.12 | 857.23 | 1422.26 | 2994.41 | 3803.91 | 500.42 | 823.97 | 1355.64 | 2986.23 | 3843.58 |
| *Thickness-to-radius ratio*: $H/R_o = 1.0$ | | | | | | | | | | | |
| 1 | $4 \times 4 \times 4$ | 246.67 | 724.38 | 1341.71 | 2643.17 | 3379.09 | 262.95 | 736.69 | 1264.46 | 2672.70 | 3377.06 |
| | $4 \times 4 \times 8$ | 246.58 | 724.36 | 1341.56 | 2638.18 | 3358.02 | 262.81 | 736.65 | 1264.18 | 2668.86 | 3361.05 |
| | $6 \times 6 \times 4$ | 246.22 | 723.94 | 1340.19 | 2623.28 | 3300.10 | 262.40 | 736.27 | 1262.55 | 2651.95 | 3302.72 |
| | $6 \times 6 \times 8$ | 246.20 | 723.93 | 1340.16 | 2623.27 | 3300.05 | 262.37 | 736.26 | 1262.49 | 2651.91 | 3302.65 |
| | $8 \times 8 \times 4$ | 246.11 | 723.88 | 1339.99 | 2622.93 | 3297.16 | 262.27 | 736.22 | 1262.27 | 2651.39 | 3300.61 |
| | $8 \times 8 \times 6$ | 246.11 | 723.88 | 1339.98 | 2622.93 | 3297.15 | 262.26 | 736.22 | 1262.25 | 2651.37 | 3300.59 |
| | $12 \times 12 \times 8$ | 246.05 | 723.86 | 1339.90 | 2622.90 | 3296.92 | 262.20 | 736.21 | 1262.13 | 2651.23 | 3300.43 |
| 2 | $4 \times 4 \times 4$ | 1174.72 | 1392.12 | 1814.52 | 3102.73 | 3886.21 | 1005.99 | 1174.12 | 1678.41 | 3092.34 | 3869.80 |
| | $4 \times 4 \times 8$ | 1174.71 | 1392.07 | 1814.29 | 3099.58 | 3878.86 | 1005.97 | 1174.04 | 1678.09 | 3087.09 | 3859.54 |
| | $6 \times 6 \times 4$ | 1174.33 | 1391.37 | 1811.39 | 3018.92 | 3743.01 | 1005.78 | 1173.25 | 1674.79 | 3054.27 | 3797.39 |
| | $6 \times 6 \times 8$ | 1174.33 | 1391.36 | 1811.37 | 3018.90 | 3742.99 | 1005.78 | 1173.23 | 1674.74 | 3054.24 | 3797.37 |
| | $8 \times 8 \times 4$ | 1174.33 | 1391.29 | 1811.13 | 3016.89 | 3741.60 | 1005.78 | 1173.14 | 1674.45 | 3053.36 | 3796.11 |
| | $8 \times 8 \times 6$ | 1174.33 | 1391.29 | 1811.12 | 3016.89 | 3741.60 | 1005.78 | 1173.13 | 1674.43 | 3053.36 | 3796.11 |
| | $12 \times 12 \times 8$ | 1174.33 | 1391.25 | 1811.02 | 3016.87 | 3741.54 | 1005.78 | 1173.08 | 1674.30 | 3053.34 | 3796.09 |