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A micro-scale modeling of Kirchhoff plate based on modified strain-gradient elasticity theory

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ABSTRACT

A Kirchhoff micro-plate model is presented based on the modified strain gradient elasticity theory to capture size effects, in contrast with the classical plate theory. The analysis is general and can be reduced to the modified couple stress plate model or classical plate model once two or all material length scale parameters in the theory are set zero respectively. Governing equation and boundary conditions of an isotropic rectangular micro-plate are derived using minimum potential energy principle. Various boundary conditions including simply supported and clamped edges are covered by the analysis. The extended Kantorovich method (EKM) which is an accurate approximate closed-form solution is applied to solve the resulting sixth order boundary value problem. Application of EKM to the partial differential equation (PDE) yields two ordinary differential equations (ODEs) in the independent *x* and *y* coordinates. The resulted ODEs are solved in an iterative manner. Exact closed-form solutions are presented for both ODEs in all of the iteration. It is shown that the method provides accurate predictions with very fast convergence. Numerical results reveal that the differences between the deflection predicted by the modified strain gradient model, the couple stress model and the classical model are large when the plate thickness is small and comparable to the material length scale parameters. However, the differences decrease with increasing the plate thickness. Validation of the presented EKM solution shows good agreement with available literature.

1. Introduction

It has been experimentally demonstrated that the micro scale structures are size-dependent. For example, it has been observed in some metals which are deformed plastically (Guo et al., 2005; Poole et al., 1996). In the micro-torsion test, Fleck et al. (1992) observed that the torsional hardening of thin copper wires increases when the wires diameter decreases. Also researchers have proven size-dependent behavior in some polymers. For instance, Chong and Lam (1999) observed strong size-dependency in epoxy and Lam et al. (2003) investigated size-dependency in epoxy polymeric beams and their results show a significant enhancement of normalized bending rigidity as the thickness of the beam decreases. In the micro-bending test of polypropylene micro-cantilevers, McFarland and Colton (2005) showed a significant difference between their results and values predicted by the classical theory of beam. The aforementioned experimental works reveal that the intrinsic behavior of some materials is size-dependent and the classical theory cannot predict reliable results due to lack of material length scale parameters while the size of structures is at micron-scale. Consequently, some higher-order theories have been proposed to take into account the size effect in which constitutive equations involve length scale parameters as well as classical Lame's constants.

One of the higher-order continuum theories is classical couple stress theory proposed by some investigators such as Toupin (1962), Mindlin and Tiersten (1962) and Koiter (1964). The theory introduces two material length scale parameters for an isotropic elastic material. The classical couple stress theory has been employed in some static and dynamic problems (Zhou and Li, 2001; Kang and Xi, 2007). Yang et al. (2002) suggested a modified couple stress theory in which a higher-order equilibrium equation, i.e. the equilibrium equation of couple of couples, is considered. As a result, the couple stress tensor should be symmetric and only symmetric part of rotation gradient tensor contributes to storage of elastic energy. Therefore, one material length scale parameter associated with the symmetric rotation gradient tensor is only included in constitutive equations in addition to two classical constants. The theory has been applied to study static and dynamic behavior of size-dependent Bernoulli–Euler and Timoshenko beam models by

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some researchers such as Park and Gao (2006), Kong et al. (2008), Ma et al. (2008), Asghari et al. (2010, 2011) and Reddy (2011). In the analysis of plates employing the modified couple stress theory, Tsiatas (2009) derived the governing equation of Kirchhoff plate with the most general form of boundary conditions and Jomehzadeh et al. (2011) studied the size-dependent vibration analysis of Kirchhoff plate.

Another higher-order continuum theory has been developed by Mindlin (1965) in which strain energy is considered as a function of first and second-order gradients of strain tensor. In a normal case, the theory involving only first-order gradient of strain tensor introduces five new constants as well as Lame's constants for an isotropic linear elastic material (Mindlin and Eshel, 1968). Altan and Aifantis (1992) proposed a simplified strain gradient theory involving only one new constant. Lazopoulos (2004) formulated a geometrically nonlinear size-dependent plate based on the simplified strain gradient elasticity theory. Fleck and Hutchinson (1993, 1997 and 2001) reformulated the Mindlin's theory and called it the strain gradient theory. Lam et al. (2003) utilizing the higher-order equilibrium equation suggested by Yang et al. (2002) modified the strain gradient elasticity theory. The theory involves three material length scale parameters corresponding to the dilatation gradient tensor, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor. The higher-order stresses are defined as the work-conjugate to the higher-order deformation metrics. It should be noted that the modified strain gradient elasticity theory can be reduced to the modified couple stress theory if two of the three material length scale parameters are taken to be zero. In other words, the modified couple stress theory is a special case of the modified strain gradient elasticity theory. The modified strain gradient elasticity has been utilized to investigate the static and dynamic response of size-dependent Bernoulli–Euler and Timoshenko beam models by some researchers such as Kong et al. (2009) and Wang et al. (2010). Buckling of axially loaded micro-scaled beams based on both of the modified couple stress theory and the modified strain gradient elasticity theory has been studied by Akgoz and Civalek (2011). Based on the simplified form of the Mindlin's strain gradient theory, a variational analysis of both rectangular and circular plated has been carried out by Papargyri-Beskou et al. (2010). Moreover, a new formulation based on the modified strain gradient elasticity theory has been developed by Wang et al. (2011) for simply supported plates. However, two misconceptions have occurred in the study concerning stress–strain relation and also extracting boundary conditions. It should be noted that the proper boundary conditions, which are derived in the presented work, are not satisfied by the double Fourier' series assumed in the Eq. (33) of the paper (Wang et al., 2011) for the static and dynamic analysis. Therefore the obtained results in both of the static and dynamic analysis would not be correct, naturally.

On the other hand, in the categories of numerical procedures, the Extended Kantorovich Method (EKM) has been first introduced by Kerr (1969) using the idea of the Kantorovich method to obtain highly accurate closed-form solution for torsion of prismatic bars with rectangular cross-section. Since then, EKM has been extensively used in many applications. For instance, one is referred to eigenvalue problems (Kerr, 1969), buckling (Yuan and Jin, 1998) and free vibrations (Dalaei and Kerr, 1996) of thin rectangular plates, bending of thick rectangular isotropic (Aghdam et al., 1996; Yuan et al., 1998) and orthotropic (Aghdam and Falahatgar, 2003) plates, free-edge strength analysis (Kim et al., 2000), vibration of variable thickness plates (Shufrin and Eisenberger, 2006) and buckling of symmetrically laminated plates (Ungbhakorn and Singhatanadgid, 2006). Although the extended Kantorovich method is based on the variational principle, it has been shown that initial guess functions are not required to satisfy the boundary conditions (Kerr and Alexander, 1961; Dalaei and Kerr, 1995; Aghdam et al., 1996). Utilizing the proposed method reduces the problem of solving a partial differential equation to a set of ordinary differential equations in the x and y directions. Iterative scheme of the method forces the solution to satisfy all boundary conditions. These two features make the EKM more appropriate than the traditional weighted residual methods such as Galerkin or Ritz method. Furthermore, the strain gradient plate models are described by a sixth order differential equation. Thus, the FEM conformity requirements demand elements of C^2 continuity which makes FEM method tedious and impractical for the problem.

The object of the present work is to provide a solution for bending analysis of a rectangular micro scale Kirchhoff plate using the modified strain gradient elasticity theory and variational principle. For this purpose, a highly accurate method, i.e. the EKM is adopted to solve the energy based derived six order PDE together with the appropriate boundary conditions. The outline of this paper is organized as follows. In Section 2, the variational formulation of the micro scale Kirchhoff plate based on the strain gradient elasticity theory is in detail deduced using the minimum potential energy principle. Then governing equation and boundary conditions are obtained simultaneously. In Section 3, the extended Kantorovich method is implemented. Subsequently, in Section4 the static bending problem for both simply supported and clamped boundary conditions is solved and numerical results of the current Kirchhoff plate model are compared with both of the classical and modified couple stress model. Validation of the presented EKM is also carried out via the available literature. Finally, some conclusions are summarized in Section 5.

2. Governing equation of micro plate

The strain gradient elasticity theory introduces dilatation gradient tensor and the deviatoric stretch gradient tensor as well as the symmetric rotation gradient. The strain energy U for an isotropic linear elastic material occupying region V based on the modified strain gradient elasticity theory is written as (Lam et al., 2003)

$$U = \frac{1}{2} \int_V \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + m_{ij}\chi^S_{ij} \right) \mathrm{d}v \tag{1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

$$\gamma_i = \varepsilon_{mm,i} \tag{3}$$

$$\eta^{(1)}_{ijk} = \eta^S_{ijk} - \frac{1}{5}\left(\delta_{ij}\eta^S_{mmk} + \delta_{jk}\eta^S_{mmi} + \delta_{ki}\eta^S_{mmj}\right) \tag{4}$$

$$\chi^S_{ij} = \frac{1}{4}\left(e_{imn}u_{n,mj} + e_{jmn}u_{n,mi}\right) \tag{5}$$

in which comma indicates partial derivative and u_i is the displacement vector, ε_{ij} is the strain tensor, γ_i is the dilatation gradient vector, $\eta^{(1)}_{ijk}$ is the deviatoric stretch gradient tensor, χ^S_{ij} is the symmetric rotation gradient tensor, δ_{ij} is the Kronocker delta, e_{ijk} is the permutation symbol and η^S_{ijk} is the symmetric part of second-order displacement gradient tensor defined by

$$\eta^S_{ijk} = \frac{1}{3}\left(u_{i,jk} + u_{j,ki} + u_{k,ij}\right) \tag{6}$$