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Dynamic eigenvector problem in thermoelectroelasticity of dissipative ionic crystals

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ABSTRACT

A two-dimensional problem in the Stroh formalism is derived for the continuum theory of thermoelectroelasticity with polarization gradients. Dissipative effects are accounted for, according to a constitutive model outlined in previous works. The eigenvector problem is studied in the frequency domain to obtain a representation of the solution in terms of two classes of modes corresponding to opposite signs of imaginary part of the eigenvalues. Impedance and admittance tensors are exploited to express the energy flux of the thermoelectroelastic transformed field across an interface $\mathcal{S}$. The compatibility conditions at $\mathcal{S}$ are also derived. The eigenvector equations are then rewritten in the time domain to obtain two convolution-type integral equations for the Hilbert transforms of the real fields corresponding to each mode.

1. Introduction

The electromechanical continuum theory of ionic crystals has received a noticeable improvement by the works of Mindlin (1968, 1969) in the late sixties. The main feature of the original approach consists in accounting for the polarization gradient in the constitutive equations in order to describe electroelastic couplings, just at a linear level, also in polarizable crystals which do not allow for the piezoelectric effect. A lattice's dynamic derivation of electroelastic coupling in alkali halide has supported this point of view showing that, in the long-wave approximation, contributions due to polarization gradient arise in the linearized model as a consequence of shell–shell and core–core interactions between the lattice's constituents (Askar et al., 1970; Askar and Lee, 1974).

Some application of this theory have been developed in the past, concerning both static and dynamic specific problems (see Mindlin, 1969; Maugin, 1988 and references therein, and, more recently, Nowacki, 2004). It is worth remarking that theories of electroelastic media which include strain gradients in the constitutive equations have been recently investigated to account for the so called "flexoelectric effect" which consists in the converse effect, where polarization arises due to a strain gradient (Maranganti et al., 2006; Majdoub et al., 2008). The interest into both direct and converse effects is motivated by their suitability to account for a noticeable electromechanical coupling in thin structures and at interfaces or surfaces.

A comprehensive theoretical approach of the continuum theory of polarizable crystals was given by Maugin (1988). He derived a non-linear theory including electromagnetic and thermal coupling, also accounting for polarization gradient and polarization inertia. Some improvements of the general theory have been recently suggested introducing the point of view of internal variables to model dissipation in accordance with the second law of thermodynamics (Romeo, 2007).

In this paper we formulate a two-dimensional vector problem in Stroh (1962) formalism, rewriting the governing equations of the linear theory of continuum thermoelasticity of polarizable dielectric solids in the form of an eigenvector equation for a sixteen components field. We allow for non-stationary solutions and use the Fourier transform to state an eigenvector problem in the frequency domain. The set of balance and constitutive equations is given in Section 2 according to the Maugin's approach. Here, looking at the Stroh-type formulation, we write the constitutive equation for the shell–shell interaction tensor as a purely homogeneous equation, accounting for the polarization effects at the free surface within the boundary conditions. The dissipative effects are modeled via internal variables according to Romeo (2007). These variables are then eliminated in writing the governing equations for the transformed field, as shown in Section 3. In Section 4 we show how the integral formalism (Lothe and Barnett, 1976a,b) can be applied in the frequency domain, and give a detailed discussion about the splitting of the solution to the eigenvector problem in two classes of modes corresponding to different signs of the imaginary part of the eigenvalues. The impedance and admittance tensors are used to express the energy flux at interfaces and the

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specific role of the polarization constraint at the free surface is outlined in Section 5. Owing to its relevance in the theory of surface and interfacial wave propagation, in Section 6 we give the compatibility conditions at an interface $\mathcal{S}$ separating two different polarizable crystals. These conditions are directly obtained by the continuity of the pertinent physical fields at $\mathcal{S}$. Finally, in Section 7 we rewrite the integrated eigenvector problem in the time domain, thus arriving at a couple of convolution-type integral equations for the Hilbert transform of the real physical field.

2. Thermoelectroelastic continuum model for ionic crystals

Here we summarize the governing equations of the continuum model of thermoelectroelastic polarizable anisotropic media (Maugin, 1988, ch. 7). They are characterized by the dependence of the constitutive functions from the polarization gradient and the presence of the polarization inertia in the balance equation for polarization.

We denote by ρ the mass density in the current configuration $\mathcal{B}_t$ of an arbitrary portion of the continuum and by $\mathbf{u}$, $\mathbf{p}$, ε and $\mathbf{q}$, respectively, the mechanical displacement, the polarization vector, the energy density and the heat flux. Adopting the quasi-static hypothesis, the electric field will be given by $-\nabla\varphi$, where φ is the electric potential. Looking at the linearized model of the general non-linear theory, in absence of external mechanical body forces and heat supplies, we write the balance equations in $\mathcal{B}_t$ as (Romeo, 2007)

$$\rho\ddot{\mathbf{u}} = \nabla\cdot T, \tag{2.1}$$

$$\delta\ddot{\mathbf{p}} = -\nabla\phi + \mathfrak{e} + \frac{1}{\rho}\nabla\cdot\mathfrak{E}, \tag{2.2}$$

$$\rho\dot{\varepsilon} = -\nabla\cdot\mathbf{q}, \tag{2.3}$$

where δ is the polarization inertia per unit mass. The quantities $\mathbf{T}$, $\mathfrak{e}$ and $\mathfrak{E}$ represent, respectively, the (Cauchy) stress tensor, the local electric field and the shell–shell or core–core interaction tensor due to the effects of polarization and its gradient. According to the divergence free condition for the electric displacement

$$\mathbf{d} = -\nabla\phi + \mathbf{p}, \tag{2.4}$$

we also have

$$\nabla\cdot\mathbf{p} - \Delta\phi = 0, \tag{2.5}$$

Denoting by $\mathbf{n}$ the outward normal to the boundary $\partial\mathcal{B}_t$ we have

$$\mathbf{n}\mathbf{T} = \mathbf{t}^{(n)}, \tag{2.6}$$

$$\frac{1}{\rho}\mathbf{n}\mathfrak{E} = \boldsymbol{\pi}^{(s)} - \mathbf{b}^{(0)}, \tag{2.7}$$

$$\mathbf{n}\cdot\mathbf{q} = q_n, \tag{2.8}$$

$$\mathbf{n}\cdot\mathbf{p} - \mathbf{n}\cdot\nabla\phi = d_n, \tag{2.9}$$

at any point in $\partial\mathcal{B}_t$. $\mathbf{t}^{(\mathbf{n})}$, q_{n} and d_{n} represent, respectively, the mechanical traction acting at the boundary, the heat flux across $\partial\mathcal{B}_t$ and the normal component of the electric displacement at $\partial\mathcal{B}_t$, while $\boldsymbol{\pi}^{(s)}$,$-\mathbf{b}^{(0)}$ are, respectively, the surface density of (possible) electric dipoles and the intrinsic "polarization traction" on $\partial\mathcal{B}_t$ (cf. Romeo, 2008). The last quantity is a constitutive parameter of the polarizable continuum, which depends on the microscopic structure of the crystal lattice (Askar and Lee, 1974). According to the previous equations, the energy flux vector of the thermoelectroelastic field can be written as

$$\mathbf{J} = -\mathbf{T}\dot{\mathbf{u}} - \frac{1}{\rho}\mathfrak{E}\dot{\mathbf{p}} + \phi\dot{\mathbf{d}} + \mathbf{q}. \tag{2.10}$$

This expression is a consequence of the energy balance for the model at hand (Maugin, 1988). It reduces to the usual energy flux vector of linear electroelasticity if the effects of polarization gradient and heat conduction are discarded (see for example Auld, 1990).

In Romeo (2007) it has been shown that an effective description of dissipative effects in the thermoelectroelastic model with polarization gradients can be obtained by introducing a symmetric second order tensor $\boldsymbol{\Omega}$ and a vector $\boldsymbol{\chi}$ which play the role of internal variables obeying a couple of supplementary suitable evolution equations. The constitutive functions for $\mathbf{T}$, $\mathfrak{e}$ and $\mathfrak{E}$ also depends on Ω and χ. In the linear case we have

$$\mathbf{T} = \mathbb{U}^{\eta\eta}\boldsymbol{\eta} + \mathbb{U}^{\eta\mathbf{p}}\mathbf{p} + \mathbb{U}^{\eta\Pi}\nabla\mathbf{p} + \mathbb{U}^{\Omega\Omega}\boldsymbol{\Omega} + \mathbb{U}^{\Omega\chi}\boldsymbol{\chi}, \tag{2.11}$$

$$\mathfrak{e} = -\left(\mathbb{U}^{\mathbf{p}\eta}\boldsymbol{\eta} + \mathbb{U}^{\mathbf{pp}}\mathbf{p} + \mathbb{U}^{\mathbf{p}\Pi}\nabla\mathbf{p}\right), \tag{2.12}$$

$$\frac{1}{\rho}\mathfrak{E} = \mathbb{U}^{\Pi\eta}\boldsymbol{\eta} + \mathbb{U}^{\Pi\mathbf{p}}\mathbf{p} + \mathbb{U}^{\Pi\Pi}\nabla\mathbf{p} + \mathbb{N}^{t}\left(\mathbb{U}^{\Omega\Omega}\boldsymbol{\Omega} + \mathbb{U}^{\Omega\chi}\boldsymbol{\chi}\right) \tag{2.13}$$

$$\mathbf{q} = -\left(\mathbb{U}^{\chi\Omega}\boldsymbol{\Omega} + \mathbb{U}^{\chi\chi}\boldsymbol{\chi}\right)\mathbb{R}, \tag{2.14}$$

where $\boldsymbol{\eta} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$ is the infinitesimal strain tensor and where $\mathbb{R}$ and $\mathbb{N}$ are non-singular, respectively, second and fourth order tensors. The tensor coefficients $\mathbb{U}$ are taken to be constant and comply with the properties

$$\mathbb{U}^{\mathbf{ab}} = \left(\mathbb{U}^{\mathbf{ba}}\right)^{t},$$

where the superimposed t denotes transposition with respect to the sets of tensorial order of $\mathbf{a}$ and $\mathbf{b}$. For instance

$$\mathbb{U}^{\Pi\mathbf{p}}_{ijk} = \mathbb{U}^{\mathbf{p}\Pi}_{kij}.$$

Within the same approximation, the evolution equations for the internal variables take the form

$$\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\eta}} + \mathbb{N}\nabla\dot{\mathbf{p}} + \gamma_{\boldsymbol{\Omega}}\boldsymbol{\Omega}, \tag{2.15}$$

$$\dot{\boldsymbol{\chi}} = \alpha\mathbb{R}\nabla\varepsilon + \gamma_{\boldsymbol{\chi}}\boldsymbol{\chi}, \tag{2.16}$$

where $\gamma_{\boldsymbol{\Omega}}$,$\gamma_{\boldsymbol{\chi}}$ are real negative parameters and $\alpha = -\mathrm{d}^2\psi/\mathrm{d}\theta^2|_0\theta_0$. Here ψ is the free energy density and θ_0 is the thermodynamic temperature of the unperturbed continuum. Since $-\psi$ is a convex function of θ, α turns out to be a real positive parameter. Also, as a consequence of the second law of thermodynamics, the block matrix

$$\begin{pmatrix} \mathbb{U}^{\Omega\Omega} & \mathbb{U}^{\Omega\chi} \\ \mathbb{U}^{\chi\Omega} & \mathbb{U}^{\chi\chi} \end{pmatrix},$$

turns out to be positive definite. The same property is supposed to hold for the block matrix (cf. Romeo, 2008)

$$\begin{pmatrix} \mathbb{U}^{\eta\eta} & \mathbb{U}^{\eta\mathbf{p}} & \mathbb{U}^{\eta\Pi} \\ \mathbb{U}^{\mathbf{p}\eta} & \mathbb{U}^{\mathbf{pp}} & \mathbb{U}^{\mathbf{p}\Pi} \\ \mathbb{U}^{\Pi\eta} & \mathbb{U}^{\Pi\mathbf{p}} & \mathbb{U}^{\Pi\Pi} \end{pmatrix}.$$