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Potentials for the modified Cam-Clay model

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ABSTRACT

Energy and dissipation pseudo-potentials are employed to derive constitutive relationships, in the context of thermodynamic concepts, for the widely used Modified Cam-Clay (MCC) model for soil mechanics. A variational formulation of the MCC evolution equations is proposed in this paper. Since plastic collapse of MCC soils cannot be embedded in the classical limit analysis theory, finding the critical amplification of the load that produces plastic collapse is formulated in the form of a system of equations and inequalities. Then, a mixed minimization principle is proposed for the plastic collapse analysis of MCC soils. This principle is obtained by the application of the variational formulation for the flow law introduced in the first part of the article.

1. Introduction

The Modified Cam-Clay (MCC) model by Roscoe and Burland (1968) is an important conceptual framework for constitutive behavior in soil mechanics (Wood, 1990; Houlsby and Puzrin, 2006; Ulm and Coussy, 2003; Borja and Lee, 1990; Vaunat et al., 2000; Einav et al., 2007). It has been widely used in its simpler, classical form and also in more sophisticated versions that extend its application to a broader variety of situations.

This paper is focused on the formal structure of the classical MCC model as derived from the energy and dissipation potentials based on thermodynamics.

The systematic application of thermodynamic concepts to the formulation of constitutive models of inelastic solids has reached a sophisticated stage due to pioneering work and many outstanding contributions. In a simplified description of this research field, thermodynamic principles are invoked to find energy and dissipation functionals that are used to derive state and evolution relations in the form of potential laws.

Among many other contributions to the thermodynamic formulation of constitutive equations, we cite first the work of Ziegler (1983). Halphen and Nguyen (1975) (see also Nguyen, 2000) proposed a framework, the Standard Generalized Materials (GSM), which is a widely referenced approach to material behavior modeling. G. T. Houlsby and coworkers (Collins et al., 1997; Houlsby and Puzrin, 2006) contributed with the concept of Hyperplasticity, which is a complete methodology to consistently develop constitutive equations based on thermodynamic principles. Rich and rigorous contributions to thermomechanics with internal variables are also found in Maugin (1992), Han and Reddy (1999) and Šilhavý (1997).

We use an extension of the concept of potential laws proposed by de Saxcé (1992), based on a special class of functions, named *bipotentials* by de Saxcé. These functions are, by definition, separately convex in two independent variables and bounded below by the scalar product of the variables. Generalized potential laws are obtained, in direct and inverse forms, by partial subdifferentiation (Rockafellar, 1970). This gives implicit constitutive equations, instead of the usual explicit direct and inverse relations. Accordingly, the class of material represented by this kind of constitutive equations is called Implicit Standard Materials (ISM) (de Saxcé, 1995; de Saxcé and Bousshine, 1998; Hjiaj, 1999; Bodovillé, 2001). It is a generalization of GSM.

This paper contains a unified derivation of energy and dissipation potentials for the MCC model, beginning with some known potentials.

We propose a mixed approach to the evolution relations of MCC materials. Throughout this paper, the expressions mixed formulation and mixed potential are used to emphasize that generalized stresses and plastic strains (or strain rates) are taken together as a pair of independent variables. In the framework of this mixed approach, we propose a minimization principle for the MCC flow law.

The mixed variational principles for MCC are applied to the analysis of plastic collapse of soils. The expression *plastic collapse*

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identifies the phenomenon of incipient unbounded plastic deformation of a body under constant loading and constant stress distribution. This is a critical state in soil mechanics. In the present context, this concept must be carefully distinguished from the case when a limit load eventually exists. Indeed, an incremental analysis of a simple system, for instance the conventional drained triaxial compression shown by Wood (1990, p. 123) and de Borst and Groen (2000, p. 37), demonstrates that the plastic collapse of MCC soils can effectively occur, in a proportional loading program, for a loading factor lower than the maximum amplification of loads previously attained. In triaxial compression this happens when heavily overconsolidated soil is sheared; we also discuss this example in Section 5.1 (see Fig. 1).

Plastic collapse of MCC soils is the subject of Section 5, where we consider: (i) a formulation of plastic collapse analysis in terms of a system of equations and inequalities and (ii) a mixed minimization principle whose solution is precisely related to the plastic collapse solution by a formal proposition, which is proven with the variational tools developed in previous sections.

2. Potentials for elastoplastic materials

In this section, we briefly introduce the basic notation used in the paper and some general potentials for the elastoplastic material constitutive equations.

2.1. State variables and energy potentials

In an isothermal process the second law of thermodynamics can be written as (Lubliner, 1972; Halphen and Nguyen, 1975; Houlsby and Puzrin, 2006; Han and Reddy, 1999)

$$D = \boldsymbol{\sigma}\cdot\boldsymbol{d} - \dot{W} \geq 0 \tag{1}$$

where D denotes specific dissipation, W is the Helmholtz free energy, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{d} := \dot{\boldsymbol{\varepsilon}}$ is the strain rate tensor and superposed dot denotes time derivative.

Let $\boldsymbol{\varepsilon}^p$ denote the plastic deformation and $\boldsymbol{\beta}$ the strain-like internal variable associated with hardening. We use this simplified notation although it is often the case that a list of strain-like scalars or tensors $\{\boldsymbol{\beta}^i; i = 1 : n_h\}$ is needed in order to represent different hardening mechanisms (Maugin, 1992, pp. 276–280; Han and Reddy, 1999, p. 52; Nguyen, 2000, 2003, p. 73, 82). The plastic strain is a special internal variable that is identified later. The internal variable $\boldsymbol{\beta}$ is not present in ideal plasticity. The generalized internal variable is defined as

$$\boldsymbol{\alpha} := (\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{2}$$

Then, the constitutive equations read as follows

$$W = \tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{3}$$

We assume that $\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$ is convex and differentiable in $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})$ and strictly convex in $\boldsymbol{\varepsilon}$ (for any fixed $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{\beta}$). Its partial Legendre-Fenchel conjugate function (Rockafellar, 1970, p. 104; Hiriart-Urruty and Lemaréchal, 1993b, p. 37; Maugin, 1992, p. 288; Han and Reddy, 1999, p. 75)

$$\tilde{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) := \sup_{\boldsymbol{\varepsilon}} \left[\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} - \tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta})\right] \tag{4}$$

is the negative of conventional Gibbs free energy, with strain-like parameters. It is convex with respect to $\boldsymbol{\sigma}$ (as are all conjugate functions) and differentiable in $\boldsymbol{\sigma}$ because $\tilde{W}$ is strictly convex in $\boldsymbol{\varepsilon}$ (Hiriart-Urruty and Lemaréchal, 1993b, pp. 38, 79).

The following state law, in direct and inverse form, is deduced from thermodynamic principles (Han and Reddy, 1999, pp. 50–52)

$$\boldsymbol{\sigma} = \nabla_{\boldsymbol{\varepsilon}}\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \Leftrightarrow \boldsymbol{\varepsilon} = \nabla_{\boldsymbol{\sigma}}\tilde{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{5}$$

with $\nabla_{\boldsymbol{\varepsilon}}\tilde{W}$ denoting the partial gradient of $\tilde{W}$ with respect to $\boldsymbol{\varepsilon}$ and $\nabla_{\boldsymbol{\sigma}}\tilde{G}$ being defined likewise.

We restrict ourselves to consider uncoupled materials, that is, materials whose tangent elastic compliance $\nabla_{\boldsymbol{\sigma}\boldsymbol{\sigma}}\tilde{G}$ (second partial gradient with respect to $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}$) is independent of the internal variables. As proven in Collins et al. (1997, pp. 1979–1981) and Han and Reddy (1999, pp. 50–52) (cf. Ulm and Coussy, 2003) this hypothesis leads to additive decomposition of strain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e(\boldsymbol{\sigma}) + \boldsymbol{\varepsilon}^p \tag{6}$$

and also to the following form for the Helmholtz free energy

$$\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{7}$$

We suppose that the elastic potential W^e is strictly convex and differentiable and that the plastic potential W^h is convex and differentiable (cf. Nguyen, 2006, p. 83).

The state equation (5) now gives

$$\boldsymbol{\sigma} = \nabla W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{8}$$

and the thermodynamic forces conjugate to $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{\beta}$ are defined as

$$\boldsymbol{\kappa} := -\nabla_{\boldsymbol{\varepsilon}^p}\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = \nabla W^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) - \nabla_{\boldsymbol{\varepsilon}^p}W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{9}$$

$$\boldsymbol{A} := -\nabla_{\boldsymbol{\beta}}\tilde{W}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) = -\nabla_{\boldsymbol{\beta}}W^h(\boldsymbol{\varepsilon}^p, \boldsymbol{\beta}) \tag{10}$$

According to the definitions (5), (9) and (10) it holds that $\dot{W} = \boldsymbol{\sigma}\cdot\boldsymbol{d} - \boldsymbol{\kappa}\cdot\boldsymbol{d}^p - \boldsymbol{A}\cdot\dot{\boldsymbol{\beta}}$. Therefore, the second law of thermodynamics (1) implies the following constraint

$$D = \boldsymbol{\mathcal{A}}\cdot\dot{\boldsymbol{\alpha}} := \boldsymbol{\kappa}\cdot\boldsymbol{d}^p + \boldsymbol{A}\cdot\dot{\boldsymbol{\beta}} \geq 0 \tag{11}$$

where the generalized internal force and flux are

$$\boldsymbol{\mathcal{A}} := (\boldsymbol{\kappa}, \boldsymbol{A}) \quad \dot{\boldsymbol{\alpha}} := \left(\boldsymbol{d}^p, \dot{\boldsymbol{\beta}}\right) \tag{12}$$

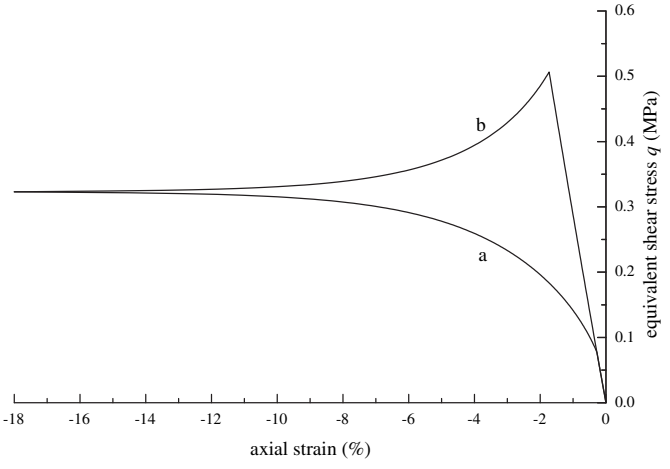

Fig. 1. Equivalent shear stress q versus axial strain ε_z in a simulated triaxial compression test with cell pressure $p_0 = 0.2$ MPa. Curve (a) corresponds to a lightly overconsolidated soil with OCR = 1.25 and (b) refers to a heavily overconsolidated soil with OCR = 5. Both are exact solutions for an MCC material with $G = 11.54$ MPa, $M = 1.05$, $\bar{\lambda} = 0.032$ and $\bar{\kappa} = 0.013$.