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Mixed method and convex optimization for limit analysis of homogeneous Gurson materials: a kinematical approach

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ABSTRACT

A fully kinematical, mixed finite element approach based on a recent interior point method for convex optimization is proposed to solve the limit analysis problem involving homogeneous Gurson materials. It uses continuous or discontinuous quadratic velocity fields as virtual variables, with no hypothesis on a stress field. Its modus operandi is deduced from the Karush–Kuhn–Tucker optimality conditions of the mathematical problem, providing an example of cross-fertilization between mechanics and mathematical programming. This method is used to solve two classical problems for the von Mises plasticity criterion as a test case, and for the Gurson criterion for which analytical solutions do not exist. Using only the original plasticity criterion as material data, the method proposed appears robust and efficient, providing very tight bounds on the limit loadings investigated.

1. Introduction

In the matter of ductile failure of porous materials, Gurson's plasticity criterion (Gurson, 1977) is the most widely accepted because it is based on a micro–macro approach and on the kinematical method of limit analysis (LA). Gurson's model treats a hollow von Mises sphere or cylinder with macroscopic strain imposed on the boundary. Recently, in Trillat and Pastor (2005), the Gurson model was validated as macroscopically representing a porous material with spherical cavities, using both statical and kinematical methods of limit analysis. The criterion that Gurson proposed for an isotropic matrix containing cylindrical cavities is expressed as follows, in plane strain:

$$(\sigma_x-\sigma_y)^2+(2\sigma_{xy})^2+8c^2\mathfrak{f}\cosh\frac{\sigma_x+\sigma_y}{2c}\leqslant 4c^2(1+\mathfrak{f}^2) \qquad (1)$$

where $\mathfrak{f}$ is the porosity rate of the material and c the flow stress in shear or cohesion. When $\mathfrak{f}=0$, this criterion reduces to the von Mises criterion.

On the other hand, F. Pastor and Loute recently developed in Pastor and Loute (2005) a general interior point algorithm to solve statical problems of limit analysis. These optimization problems present a linear objective function and a mix of linear and non-linear convex constraints. For problems where the plasticity criterion is the von Mises criterion, the non-linear constraints are convex quadratic inequalities, giving rise to a conic programming problem for which efficient algorithms and codes exist (Ben-Tal and Nemirovskii, 2001; MOSEK ApS, 2002). The Gurson criterion leads to convex inequality constraints, which do not fit the second-order conic programming (SOCP) formulation. Hence, an optimization solver, adapted from an algorithm presented by Vial (1994), was presented in Pastor (2001) and improved in Pastor and Loute (2005) for solving the statical problems for von Mises and Gurson materials. Henceforth, this optimization solver will be called IP-SOLVER.

The classical solution of the kinematical problem is more complex, especially in the Gurson case, because the dissipated power is not always easy to take into account. In the present paper, this difficulty is bypassed using a specific mixed finite element method. To our knowledge, the first mixed approaches were proposed by Capurso in 1971 (Capurso, 1971) and by Anderheggen and Knopfel in 1972 (Anderheggen and Knopfel, 1972) for continuous velocity fields and piecewise linear criteria. A general mixed formulation of the limit analysis problem in terms of loading parameters was also given by Radenkovic and Son in 1972 (Radenkovic and Nguyen, 1972).

On the basis of Christiansen's mixed formulation, fully investigated in Christiansen (1996), Ciria and Peraire (2004) extended in 2004 the work of Anderheggen and Knopfel (1972) to discontinuous linear velocity in plane strain, applying it to von Mises materials using SOCP codes. The Anderheggen and Knopfel formulation was also extended in 2005 to discontinuous linear velocity by Krabbenhoft et al. (2005), where a velocity discontinuity segment is simulated by means of two thin finite elements. Though based on the duality properties of linear programming assumed to

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be valid in the non-linear case, this approach appears to give kinematical solutions through a somewhat complicated formulation. In any case, this formulation cannot be extended, as it is, to the quadratic case without losing its kinematical character. Another direct formulation, which uses convexity properties to strictly upper bound the dissipated power on the linear discontinuity surfaces was proposed in the mixed approach of Pastor et al. (2006a), providing rigorous kinematical solutions.

An extension to take into account discontinuous quadratic velocity fields was first presented in a summarized form in Pastor et al. (2006b, 2006c). Given in detail here, this extension is a so-called mixed finite element method, but purely kinematical. Thanks to the convex nature of the set of the plastically admissible strain rates and to the convexity of the unit dissipation functions, continuous or discontinuous quadratic velocity fields are taken into account with no hypothesis on a stress field: only a finite set of independent stress tensors appropriately located on the finite elements or on the discontinuity segments is needed. Particular attention is paid to the velocity discontinuities in order to demonstrate that a direct extrapolation from the linear case cannot give rigorous upper bounds without additional conditions. The proposed formulation fulfills all these requirements and is very easy to implement.

We chose to test the method first on the problem of a notched bar under lateral tensile stress, often used in the literature, and on the problem of a bar compressed between rough rigid plates. By using IP-SOLVER, continuous and discontinuous quadratic velocity fields are analyzed in relation with the statical values obtained as in Pastor and Loute (2005). Examples of a von Mises material (for validation) and a Gurson material are tested in both problems. In all tests, the kinematical solutions come very close to the statical solutions; the Gurson results, the first ones in limit analysis – to our knowledge – also give an original set of rigorous lower/upper bound values that are also very useful for validating elastoplastic methods, for example. Somewhat unexpectedly, this test also resulted in confirming that it is highly recommended to post-verify the velocity fields given by an optimizer: in these tests, IP-SOLVER appears noticeably more reliable than commercial SOCP codes, which in principle are well suited to the von Mises material.

In the following section we review the initial formulation of the optimization method of Pastor and Loute (2005), limited only to what is needed further. Then we present the proposed kinematical mixed method, and its detailed application to two von Mises and Gurson mechanical problems.

2. Interior point method and convex optimization

In Pastor and Loute (2005), a general interior point algorithm for solving the statical problem of LA is detailed; this paper focused on solving the plane strain LA problems for both von Mises and Gurson materials. The resulting optimization problems present a linear objective function and a mix of linear and non-linear convex constraints. For problems where the plasticity criterion is the von Mises or Drucker–Prager criterion, the non-linear constraints are convex quadratic inequalities, generating SOCP problems for which efficient algorithms and codes exist. Indeed, this is not the case with the Gurson criterion for example. The general form of the optimization problems we will have to solve here is as follows:

$$\begin{aligned} &\max c^T x \\ &\text{s.t. } Ax = b, \\ &g(x) + s = 0, \quad s \geqslant 0, \end{aligned} \tag{2}$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ is the matrix of the linear constraints, $g = (g_1, \ldots, g_p)$ is a vector-valued function of p convex numerical functions g_i, and $s \in \mathbb{R}^p_+$ is the vector of slack variables associated with these convex constraints.

The "primal-dual interior point method" consists in solving, instead of the previous problem, the following one, parametrized by $\mu > 0$, the "barrier parameter":

$$\begin{aligned} &\max\left(c^T x + \mu \sum_{i=1}^{p} \ln(s_i)\right) \\ &\text{s.t. } Ax = b, \\ &g(x) + s = 0, \quad s > 0. \end{aligned} \tag{3}$$

Using the "primal-dual interior point method", the problem (2), has a solution if and only if the following conditions are satisfied:

$$\begin{aligned} &-c + A^T w + \left(\frac{\partial g}{\partial x}\right)^T y = 0, \\ &Ax - b = 0, \\ &g(x) + s = 0, \\ &Y S e = \mu e, \end{aligned} \tag{4}$$

where $w \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $e = [1 \ldots 1]^T \in \mathbb{R}^p$ and Y, S are the diagonal matrices associated with y and s, respectively; $\mu > 0$ and $s > 0$ imply $y > 0$.

For each given μ the non-linear system (4) is approximately solved by one iteration of the Newton method, thereby providing an approximate solution of the parametrized problem (3). Using a sequence of values for μ decreasing to zero, we make the latter converging to the solution of (2). Indeed, as μ approaches 0, Eqs. (4) come close to the KKT conditions for the original problem.

3. Limit analysis and variational formulation

For the sake of clarity, here we consider that the velocity fields are continuous, with discontinuous fields analyzed later.

3.1. Reminder of LA

According to Salençon (see Salençon, 1974, 1983), a stress tensor field σ is said to be statically admissible (SA) if equilibrium equations, stress vector continuity, and stress boundary conditions are verified. It is said to be plastically admissible (PA) if $f(\sigma) \leqslant 0$, where $f(\sigma)$ is the (convex) plasticity criterion of the material. A field σ that is SA and PA here will be said to be (fully) admissible.

Similarly, a strain rate tensor field v is kinematically admissible (KA) if it is derived from a continuous velocity vector field u such that the velocity boundary conditions are verified. It is said to be plastically admissible (PA) if the flow rule (5) is verified, and the fields u, v, which are KA and PA, will be called admissible in the following.

$$v = \lambda \frac{\partial f}{\partial \sigma}, \quad f(\sigma) = 0, \quad \lambda \geqslant 0. \tag{5}$$

The so-called associated flow rule (5) (or normality law) characterizes the standard material of LA. In an equivalent manner, a standard material verifies Hill's maximum work principle (MWP) (Hill, 1950), which states:

$$(\sigma - \sigma^*) : v \geqslant 0 \quad \forall \text{ PA } \sigma^* \tag{6}$$

Consequently, the *convex* unit dissipated power $\pi(v)$ is defined as:

$$\pi(v) = \sigma : v \quad \text{if (5) or (6) is verified,} \tag{7a}$$

$$\pi(v) = +\infty \quad \text{otherwise.} \tag{7b}$$