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The role of extensibility in the birth of a ruck in a rug

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ABSTRACT

Everyday experience suggests that a 'ruck' forms when the two ends of a heavy carpet or rug are brought closer together. Classical analysis, however, shows that the horizontal compressive force needed to create such a ruck should be infinite. We show that this apparent paradox is due to the assumption of inextensibility of the rug. By accounting for a finite extensibility, we show that rucks appear with a finite, non-zero end-shortening and confirm our theoretical results with simple experiments. Finally, we note that the appropriate measure of extensibility, the stretchability, is in this case not determined purely by geometry, but incorporates the mechanics of the sheet.

1. Introduction

Localized bumps (or rucks) in a carpet are an everyday annoyance but have also been used as an analogy to understand a plethora of physical phenomena involving the sliding of two solid bodies. Examples include dislocations in plastic deformation [1], Schallamach waves in rubber friction [2–4], and even slip pulses in earthquakes [5]. Recently, a number of authors have focused on the motion of such rucks, which can occur either rapidly (as when a rug is quickly shaken at one end) [6,7] or slowly (as when a ruck 'falls' down an inclined plane) [8,9]. Surprisingly, however, some features of a static ruck remain poorly understood, including the conditions under which they form.

At a superficial level, the formation, or birth, of a ruck in a rug is similar to the buckling of the classic Euler elastica [10]: the two ends of a rug are brought closer together by a distance $\Delta\ell$ and a ruck forms to accommodate the resulting excess length. Unlike the Euler elastica, however, the amount of buckled material is not equal to the whole system size—some material remains in contact with the 'floor' and the arc-length of the ruck, l, is not known *a priori*.

Everyday experience suggests that l depends on the end–end compression $\Delta\ell$. To determine this relationship at the scaling level we follow Kolinski et al. [8]. For small rucks, simple geometry suggests that the height of the ruck $d \sim (l\Delta\ell)^{1/2}$. For a rug of density ρ_s, thickness t and bending rigidity B, we expect that the gravitational energy of the ruck $\sim \rho_s g t \times d \times l$, while the bending energy $\sim B(d/l^2)^2 \times l$. Balancing these energies and eliminating $d \sim (l\Delta\ell)^{1/2}$ we find that

$$\Delta\ell \sim \left(\frac{\rho_s g t}{B}\right)^2 l^7. \tag{1}$$

As expected, the width of the ruck grows with increasing end–end compression.

How much compressive *force* is required for the onset of rucking? A simple calculation [6] reveals that the compressive force T required to form a ruck of size l satisfies

$$\frac{Tl^2}{B} \approx 80.76, \tag{2}$$

with B the bending rigidity. Note that (2) is precisely the classical result for the Euler buckling of a rod of length l with clamped ends [11]. Using (1) to eliminate l from (2) in favour of $\Delta\ell$ reveals that the buckling load

$$T \sim \frac{B^{3/7}(\rho_s g t)^{4/7}}{\Delta\ell^{2/7}}, \tag{3}$$

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which is divergent in the limit of very small end-shortenings $\Delta\ell$. To form a ruck from a flat rug we must pass through arbitrarily small end-shortenings and so we are led to the paradoxical result that to do so requires an infinite compressive force!

This divergence in the compressive force needed to form a ruck has been known for almost thirty years [12]. Previous authors have attempted to explain it as a result of a breakdown of the linearized beam theory used to obtain (1) and (2) [13–15] while [16] confirmed the validity of the linearized approach and showed decisively that a critical buckling load does not exist for a perfect, infinite, continuous horizontal heavy elastica. Here, we show using a combination of theory and experiment that the paradox is resolved by incorporating the finite extensibility of the material.

A key assumption in the preceding discussion is that the object, the rug in this case, has a fixed arc length, i.e. that it is inextensible. While this is true for objects with arbitrarily small thickness, any real object has a finite, if small thickness, and so is, to a certain extent, extensible. The effect of finite extensibility on the buckling of the classical Elastica has been studied by a number of authors [17–19]. These analyses reveal that the crucial parameter governing the importance of extensibility is the ratio of the thickness, t, and total length, L_{tot}, of the beam or, equivalently, the von Kármán number, $\gamma = t^2/(12L_{\text{tot}}^2)$. However, the effect of γ on the buckling threshold and the post-buckling behaviour is, in fact fairly small [18]. An unusual exception is the vibration frequencies of some modes of buckled beams, which may be different for perfectly inextensible beams ($\gamma = 0$) and asymptotically inextensible beams ($\gamma \to 0$): the limit is singular [18].

In this context, the classic prediction that the compressive force required to give birth to a ruck diverges clearly suggests the possibility that the finite extensibility must ultimately play a role in regularizing the problem; we expect that inclusion of this effect will lead to a finite compressive force and a finite ruck size at the onset of buckling. Nevertheless, this suggestion appears not to have been made previously [12] and so we study this problem here. Using a combination of theory and tabletop experiments, we show that indeed the finite extensibility of any physical (rather than idealized) heavy elastica does give rise to a finite ruck height at birth and, hence, a finite buckling load. This is similar to the problem of the growth of a rod inside a curved cylinder where it was recently shown that finite extensibility regularizes the otherwise infinite pressure that would be applied on the cylinder prior to buckling [20]. However, unlike this related problem, and previous studies of the role of extensibility, we find that the relevant parameter governing the role of extensibility is *not* simply determined by the geometrical properties of the object (namely t and L_{tot}) but depends also on its material properties (namely its Young's modulus E and density ρ_s).

2. Theoretical analysis

2.1. Problem formulation

We model the rug as a heavy elastic sheet of solid density ρ_s, length ℓ_∞, thickness t and width b. This sheet rests on a solid horizontal surface, which we assume to be frictionless for simplicity, and is compressed by bringing the two ends a horizontal distance $\Delta\ell$ closer together (see Fig. 1(c)). The sheet can respond to this compression in two ways: by compressing along its length and by buckling out of the plane. If the sheet buckles, it does so over a finite region of width $l < \ell_\infty$ because the weight per unit area of the sheet ($\rho_s g t$) opposes the whole sheet losing contact with the surface. Assuming small transverse displacements, the profile of the sheet $y = w(x)$ satisfies the linearized heavy elastica equation [6,12]

$$B\frac{d^4w}{dx^4} + T\frac{d^2w}{dx^2} = -\rho_s g t \tag{4}$$

for $|x| < l/2$. For $l/2 \le |x| < \ell_\infty/2$ we have $w = 0$: the sheet is in contact with the substrate.

In (4), $B = Et^3/[12(1-\nu^2)]$ is the bending stiffness of the sheet per unit width, with ν the Poisson Ratio. As the problem is symmetric about $x = 0$, we consider only $0 \le x < \ell_\infty/2$, for simplicity. This gives rise to the symmetry boundary conditions

$$w'(0) = w'''(0) = 0. \tag{5}$$

Since $w = 0$ for $l/2 < |x| < \ell_\infty/2$, continuity of vertical displacement, together with force and torque balance at $x = l/2$ then give

$$w(l/2) = w'(l/2) = w''(l/2) = 0. \tag{6}$$

The system (4)–(6) is ostensibly a fourth order system with five boundary conditions. However, the size of the buckled region, l, is unknown as is the imposed compressive force T (since we are assuming that the sheet is subject to a known displacement of its ends). In fact, then, we have a sixth order problem with the final condition coming from a relationship between compression and tension. Ordinarily, one might be tempted to specify that the imposed compression should be accounted for by the out of plane displacement, so that $\Delta\ell = \int_0^{l/2}(w')^2\,dx$. However, this condition implicitly assumes that the sheet is inextensible. Instead we return to Hooke's law, which relates T to the strain, e_{xx} within the sheet

$$T = -\frac{Et}{1-\nu^2}e_{xx}. \tag{7}$$

Since the strain can be expressed in terms of the horizontal and out-of-plane displacements u and w, respectively, we then have

$$-\frac{1-\nu^2}{Et}T = e_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2. \tag{8}$$

Integrating from 0 to $\ell_\infty/2$, we find that

$$-\frac{\ell_\infty(1-\nu^2)}{2Et}T = \frac{1}{2}\int_0^{\ell_\infty/2}\left(\frac{dw}{dx}\right)^2 dx - \frac{\Delta l}{2}, \tag{9}$$

where we have used $u(\ell_\infty/2) = -\Delta\ell/2$ as the imposed displacement. (In the above derivation we have used that the compressive force T is constant; Hooke's law (7) then shows that the strain e_{xx} is also constant.)