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The effective non-linear properties of a composite coating and a composite sandwich layer☆

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ABSTRACT

Hashin–Shtrikman based bounds and estimates are obtained for the linear and non-linear effective properties of composites in the form of a thin coating or sandwich layer. It is assumed that the thickness of the layer is of the same order of magnitude as the correlation length between phases, and size effects thereby result. Boundary layers exist within the coating adjacent to the substrate and to the free surface (in the case of a coating). Attention is focused on two-dimensional problems by considering anti-plane shear of an isotropic 2-phase composite on a single-phase substrate, with microstructure prismatic along the direction of anti-plane shear.

1. Introduction

Surface coatings and embedded layers are ubiquitous in engineering components, and serve a wide range of functions from environmental protection to low friction and wear resistance. Indeed, the field of surface engineering involves the manufacture of coatings with a wide range of multifunctional properties. The coating may be stiffer (and stronger) than that of the substrate, for example the surface layer of aluminium alloys can be converted to aluminium oxide by anodisation. Or, the coating may be softer and more compliant, such as zinc-coated steel, paints, low friction polymer coatings (such as PTFE on steel or aluminium alloy) and thermal barrier coatings. A related geometry to the surface coating is the embedded layer sandwiched between two substrates. This geometry is also ubiquitous and is representative of adhesive joints, the mortar between the bricks of a building, and interphases at grain boundaries *inter alia*.

Frequently, a coating comprises a multi-phase composite with, for example, particulate reinforcement in order to increase its stiffness and strength. *The question arises: what are the effective properties of a composite coating?* A common assumption is to use the effective properties of the *bulk* composite for that of the *coating*. Whilst this assumption is accurate when the correlation length of each phase is much less than the coating thickness, it is less accurate when the two length scales are of comparable magnitude. The presence of the substrate or a free surface perturbs the stress field within the composite coating. This can be re-phrased in a more mathematical manner, as follows. The usual Hashin–Shtrikman variational approach for the bulk properties of a composite makes use of the infinite-body Green's function in order to determine the ensemble-averaged strain field in terms of a polarization in stress from one phase to the next. For the embedded layer, the infinite-body Green's function is employed, whereas for a surface coating the half-space Greens function is exploited.

The purpose of this study is to make accurate predictions for the effective properties of a *surface composite coating* or an *embedded composite* layer, taking into account the presence of the substrate of differing properties, whether *linear* or *non-linear*. Effective properties and associated bounds are generated for composite coatings and for composite sandwich layers of finite thickness, based on the Hashin–Shtrikman approach, but suitably modified to account for the presence of a free surface in the case of a coating and of substrates in the case of a sandwich layer. First, the linear properties are generated and then the method is modified to generate bounds and estimates for a non-linear composite coating. We shall limit our scoping study to two-dimensional problems by considering anti-plane shear of an isotropic 2-phase composite on a single-phase substrate, with microstructure prismatic along the direction of anti-plane shear.

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2. Statement of problem: a composite half-space in antiplane shear: the 2D linear case

We shall consider the anti-plane shear response of a coating of height h made from a random M-phase composite, adhered to a monolithic substrate of height $H >> h$ made from phase $M+1$ material. The outer top surface of the coating is subjected to a longitudinal shear traction σ_y^∞, while the base of the substrate is rigidly held without displacement, see Fig. 1a. Both the coating and substrate are initially treated as linear elastic, with the non-linear behaviour addressed in a subsequent section. The origin of a Cartesian reference frame (x,y,z) is placed on the top external surface of the coating, with the y-direction aligned with the outward normal to the external surface. Thus, the coating extends over $-h \le y \le 0$, and the underlying substrate occupies $-(H+h) \le y \le -h$. The z-axis aligns with the direction of anti-plane shear. Results will be presented in the limit $H/h \to \infty$ but the recognition that H is actually finite is needed to ensure convergence of certain integrals during the derivation.

A closely related problem is an M-phase composite layer of thickness h sandwiched between two substrates made from phase $M+1$, with the assembled stack subjected to a longitudinal shear traction σ_y^∞, see Fig. 1b. For this case, the origin of a Cartesian reference frame (x,y,z) is placed on the upper interface of the coating, such that the coating extends over $-h \le y \le 0$, as shown in the figure. In our study, we shall focus on the coating problem of Fig. 1a but shall include the analysis and results for the sandwich layer at appropriate steps in the development.

The distribution of phases within the coating is taken to be isotropic, and each phase has a linear, isotropic response. In contrast, the isotropic substrate beneath the coating is taken to be homogeneous. We seek the effective properties of the coating. The single non-vanishing displacement $u(x,y)$ is in the z-direction. The resulting (engineering) shear strain has components $e_x = u_{,x}$ and $e_y = u_{,y}$ and the work-conjugate stress has the shear components $\sigma_x \equiv \sigma_{zx}$ and $\sigma_y \equiv \sigma_{zy}$, respectively. For later convenience, a Greek suffix takes the values of x or y, and a repeated Greek suffix denotes summation, in accordance with the usual Einstein notation. For example, $u_{,\alpha}$ denotes $u_{,x}$ or $u_{,y}$; σ_α denotes σ_x or σ_y; and $u_{,\alpha\alpha}$ denotes $u_{,xx} + u_{,yy}$.

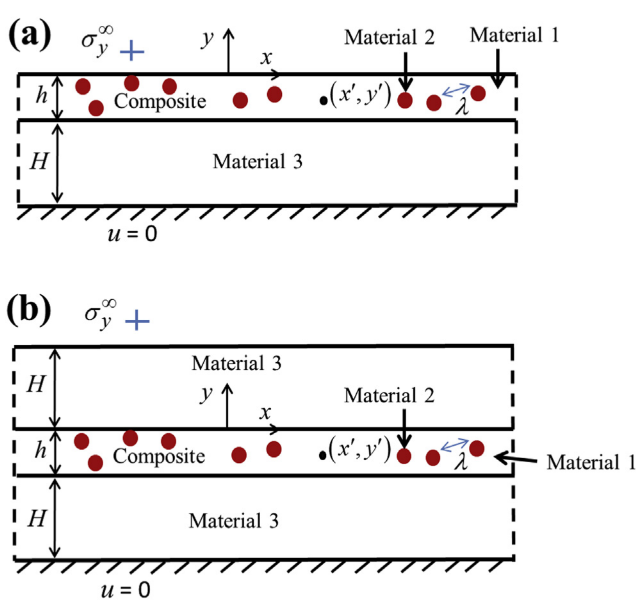

Fig. 1. (a) A coating and (b) a sandwich layer, comprising a two phase composite adhered to a substrate, under an imposed longitudinal shear traction σ_y^∞.

The coating comprises an M-phase random composite, and each phase r is isotropic and of shear modulus μ_r. The substrate is homogeneous and isotropic, and is made from phase $M+1$ of shear modulus μ_{M+1}. No variation in microstructure and material properties exists along the z-axis; recall that the applied surface shear traction σ_y^∞ is also along this direction. The stress components σ_α at a given point $\boldsymbol{x} = (x,y)$ are related to the strain components at that point according to

$$\sigma_\alpha = \mu e_\alpha = \mu u_{,\alpha} \tag{2.1}$$

where $\mu(\boldsymbol{x})$ takes the value μ_r if $\boldsymbol{x}$ lies in material of type r. Thus,

$$\mu(\boldsymbol{x}) = \sum_{r=1}^{M+1} \mu_r \chi_r(\boldsymbol{x}) \tag{2.2}$$

where the characteristic function $\chi_r(\boldsymbol{x})$ takes the value of unity if material r is at $\boldsymbol{x}$ and equals zero otherwise. We seek the overall effective response of the layer and substrate. First, we record the expressions for the bulk composite, as a benchmark.

2.1. A summary of the effective response of an M-phase composite in shear

It is instructive to compare the stiffness of the composite coating with that of the bulk composite. In order to do so, we assemble here the well-established results for the bulk response of a composite of volume fraction p_r for each phase r. For completeness, we write the elementary bounds in the above notation. The elementary bounds imply a uniform strain distribution within the coating of magnitude $e_y = \sigma_y^\infty/\overline{\mu}$ where

$$\overline{\mu} = \overline{\mu}_V \equiv \sum_{r=1}^{M} p_r \mu_r \tag{2.3}$$

for the Voigt bound, and

$$\overline{\mu} = \overline{\mu}_R \equiv \left(\sum_{r=1}^{M} p_r \mu_r^{-1} \right)^{-1} \tag{2.4}$$

for the Reuss bound.

Hashin–Shtrikman bounds and estimates for the effective shear modulus of an M-phase composite have been derived by Hill (1964, 1965) and Walpole (1969). For a comparison medium of shear modulus μ_0 the Hashin–Shtrikman estimate reads

$$\mu_{\mathrm{HS}} = \left(\sum_{r=1}^{M} \frac{p_r}{\mu_r + \mu_0} \right)^{-1} - \mu_0 \tag{2.5}$$

The Hashin–Shtrikman upper bound μ_{HS}^+ is attained by setting μ_0 to the largest value of shear modulus over all phases. Likewise, the Hashin–Shtrikman lower bound μ_{HS}^- is attained by setting μ_0 to the smallest value of shear modulus over all phases. And the self-consistent estimate μ_{SC} is obtained by identifying μ_0 with μ_{HS} in (2.5) and then solving for μ_{HS}; a convenient way to accomplish this in the general case is to employ iteration to convergence of (2.5).

Now re-write (2.5) for the case of a 2-phase composite, and assume without loss of generality that $\alpha \equiv \mu_2/\mu_1 > 1$. Then,

$$\frac{\mu_{\mathrm{HS}}^+}{\mu_1} = \frac{1 + p_1 + \alpha p_2}{1 + p_1 + (p_2/\alpha)} \tag{2.6}$$