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A numerical approach to the yield strength of shell structures

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ABSTRACT

This work investigates the formulation of lower and upper bound finite elements for the yield design (or limit analysis) of shell structures. The shell geometry is first discretized into triangular planar facets so that previously developed lower bound equilibrium and upper bound kinematic plate finite elements can be coupled to membrane elements. The other main novelty of this paper relies on the formulation of generalized strength criteria for shells in membrane-bending interaction via an implicit upscaling procedure. This formulation provides a natural strategy for constructing lower and upper bound approximations of the exact shell strength criterion and are particularly well suited for a numerical implementation using second-order cone programming tools. By combining these approximate strength criteria to the previously mentioned finite elements, rigorous lower and upper bound ultimate load estimates for shell structures can be computed very efficiently. Different numerical examples illustrate the accuracy as well as the generality and versatility of the proposed approach.

1. Introduction

Computational direct methods to estimate the ultimate load of a various range of structures using yield design (or limit analysis in the context of an elastic perfectly plastic behavior) theory have gained increasing attention in the last decades thanks to the development of efficient optimization solvers, in particular interior-points algorithm for conic programming problems. The static approach, which consists in maximizing the load multiplier over a set of statically admissible stress fields, while satisfying the local strength at each point of the structure, enables to obtain a lower bound estimate for the ultimate load. Conversely, the dual approach, namely the kinematic approach, which consists in minimizing the maximum resisting work (or plastic dissipation in the context of limit analysis) over a set of kinematically admissible virtual velocity fields, enables to obtain upper bound estimates for the ultimate load. The finite element method can be implemented in the context of both approaches using equilibrium elements for the static approach or kinematic finite elements (which may include potential discontinuities) for the kinematic approach. The subsequent optimization problems can then be solved using dedicated convex programming solvers, depending on the mathematical expression of the strength criterion.

The general framework of the yield design theory (Salençon, 1983, 2013) can be applied to a wide variety of mechanical models: 2D/3D continuous media, beam structures, plates and slabs, shells, etc. As regards the latter structures, one key issue is then to formulate a strength criterion in terms of the generalized internal forces of the considered model. From a numerical point of view, attention has also been devoted to the formulation of such criteria using conic (in particular second-order cone) constraints (Bisbos and Pardalos, 2007; Makrodimopoulos, 2010) in order to employ efficient second-order cone programming (SOCP) solvers, such as the MOSEK software package (Mosek, 2014).

Focusing more specifically on the limit analysis/yield design of shell structures, early works have been mainly dedicated to deriving analytical lower and upper bound estimates for simple structures (Hodge, 1954, 1963, 1959; Prager, 1961; Save et al., 1997). An important number of such solutions is compiled in (Save, 1995). In this context, the possible use of a generalized strength criterion for shells in membrane-bending interaction has been discussed (Ilyushin, 1956), and various approximate criteria have been proposed to simplify the analysis (Robinson, 1971). From a computational point of view, papers dedicated to a numerical implementation of limit analysis applied to such structures remain quite scarce. Limit analysis of axi-symmetric shells using non-linear programming has been proposed in (Hung et al., 1978). Shakedown analysis (extension of limit analysis to cyclic loadings) of axi-symmetric shells using a linearized yield surface has been

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investigated in (Franco and Ponter, 1997a,b) and extended in (Franco et al., 1997) including error analysis and adaptive remeshing. Sequential limit analysis has been used to investigate the post-collapse behavior of a structure by updating the geometry from using collapse mechanisms obtained from the kinematic approach. Such analyses have been performed in (Corradi and Panzeri, 2003, 2004) in the case of a von Mises criterion and in (Raithatha and Duncan, 2009) using Ilyushin approximate criterion (although some concerns with respect to the upper bound formulation have been raised in (Makrodimopoulos and Martin, 2010)). A continuum-based shell element has been developed to perform shakedown analysis on von Mises shells in (Martins et al., 2014), but due to the mixed formulation, only approximate values of the limit loads can be obtained. Upper bound limit and shakedown analysis based on the exact Ilysuhin strength criterion has been considered in (Tran et al., 2008), using a particular optimization procedure. Finally, one can mention the work of Bisbos and Papaioannou (Bisbos and Papaioannou, 2006), who used a Morley shell element and SOCP to solve problems involving Ilyushin approximate criterion.

From all these previous works, it can first be observed that the derivation of lower bound shell elements is almost non-existent, although it can be highly beneficial to obtain a reliable bracketing of the true ultimate load. Secondly, almost all existing works are focused on homogeneous metal shells, which can be modeled using the von Mises strength criterion. To the authors' opinion, this particular focus is partly due to the difficulty of deriving appropriate general strength criteria for other types of constitutive materials (reinforced concrete for example). Therefore, the present work aims at contributing to the derivation of both lower and upper bound yield design shell finite elements as well as proposing a general formulation of generalized shell strength criteria which can be used in conjunction with state-of-the-art conic programming solvers.

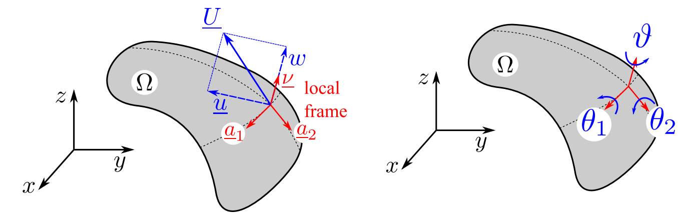
The paper is organized as follows: in Section 2, the main features of the yield design analysis of shell structures are recalled, relying on the formulation of a strength condition expressed in terms of generalized stress variables, such as membrane forces and bending moments. Section 3 advocates the use of a piecewise linearization of the shell geometry by discretizing its curved surface into triangular facets, that is by replacing the initial shell by a continuous assemblage of triangular plates, for which lower and upper bound finite element formulations have been previously developed. Section 4 is then devoted to the key issue of formulating a generalized stress-based criterion for a multilayered shell, which can be obtained from solving a yield design problem, where the distribution of material local strength properties across the shell thickness is known. The numerical performance of such an upscaling procedure is favorably compared with classical existing solutions or approximations. Finally, Section 5 presents some illustrative applications of the whole numerical procedure, where the combination of the plate finite element formulation and strength criterion approximation, in the context of the lower or upper bound approach, leads to a single global SOCP optimization problem.

2. Yield design of shells: a brief outline

2.1. General formulation using the static approach

Referring to a Cartesian orthonormal frame $(O;\underline{e}_x,\underline{e}_y,\underline{e}_z)$, the shell occupies a two-dimensional manifold Ω. The shell geometry can be described locally by a unit normal $\underline{\nu}$ and a tangent plane spanned by two unit vectors $\underline{a}_1$ and $\underline{a}_2$ (Fig. 1).

The generalized internal forces of the shell model are described by a symmetric tensor $\underline{\underline{N}} = N_{ij}\underline{a}_i \otimes \underline{a}_j$ of *membrane forces*, a symmetric tensor $\underline{\underline{M}} = M_{ij}\underline{a}_i \otimes \underline{a}_j$ of *bending moments* and a vector $\underline{V} = V_i\underline{a}_i$ of *shear forces* $(i,j = 1,2)$.



(a) Virtual velocity (b) Virtual angular velocity

Fig. 1. Shell kinematics.

Let then $G(\underline{x})$ be the *generalized strength domain* of the shell at $\underline{x}\in\Omega$, which is a convex set in the $(\underline{\underline{N}},\underline{\underline{M}},\underline{V})$ space (of dimension 8).

Assuming that the shell external loading depends upon several loading parameters collected in a vector $\underline{Q}$, the domain K of *potentially safe loads* $\underline{Q}$, as introduced in the yield design theory (Salençon, 1983, 2013), is defined as follows:

$$K = \{\underline{Q};\quad \exists(\underline{\underline{N}},\underline{\underline{M}},\underline{V})\ \text{SA with } \underline{Q} \\ \text{and } \forall\underline{x}\in\Omega \\ (\underline{\underline{N}}(\underline{x}),\underline{\underline{M}}(\underline{x}),\underline{V}(\underline{x}))\in G(\underline{x})\} \tag{1}$$

where a distribution of generalized internal forces $(\underline{\underline{N}},\underline{\underline{M}},\underline{V})$ must satisfy all local equilibrium, continuity and boundary conditions in order to be statically admissible (SA) with a given loading $\underline{Q}$.

2.2. Dual formulation using the kinematic approach

A dual definition of the domain K of potentially safe loads using the kinematic approach is classically obtained by means of the virtual work principle. In the context of a shell model, the virtual kinematics of the shell is characterized at any point $\underline{x}\in\Omega$ by (Fig. 1):

- a virtual velocity $\underline{U} = U_x\underline{e}_x + U_y\underline{e}_y + U_z\underline{e}_z$ of the particle attached to the point. It will be convenient to express $\underline{U}$ in terms of its in-plane $\underline{u} = u_i\underline{a}_i$ and out-of-plane $w\underline{\nu}$ components, so that: $\underline{U} = \underline{u} + w\underline{\nu}$.
- a virtual angular velocity $\underline{\Theta} = \Theta_x\underline{e}_x + \Theta_y\underline{e}_y + \Theta_z\underline{e}_z$, representing the rotation of the microstructure attached to the same point. Again, it will be useful to express it in terms of in-plane and out-of-plane components: $\underline{\Theta} = \underline{\theta} + \vartheta\underline{\nu}$ with $\underline{\theta} = \theta_i\underline{a}_i$.

Hence, the virtual work principle reads as:

$$(\underline{\underline{N}},\underline{\underline{M}},\underline{V})\ \text{S.A. with } \underline{Q}\Leftrightarrow \\ \forall(\underline{U},\underline{\Theta})\ \text{K.A. with } \underline{\widehat{q}}, \\ P_{(e)}(\underline{U},\underline{\Theta}) = \underline{Q}\cdot\underline{\widehat{q}} = -P_{(i)}(\underline{U},\underline{\Theta})$$

where $P_{(e)}(\underline{U},\underline{\Theta}) = \underline{Q}\cdot\underline{\widehat{q}}$ represents the virtual work of external loading ($\underline{\widehat{q}}$ being the generalized kinematic parameters associated by duality to the loading parameters $\underline{Q}$) and $P_{(i)}(\underline{U},\underline{\Theta})$ the virtual work of internal forces which can be expressed as follows:

$$P_{(i)}(\underline{U},\underline{\Theta}) = -\int_\Omega \left(\underline{\underline{N}}:\underline{\underline{\varepsilon}} + \underline{\underline{M}}:\underline{\underline{\chi}} + \underline{V}\cdot\underline{\gamma}\right)d\Omega \tag{2}$$

where $\underline{\underline{\varepsilon}}$ are the virtual *membrane strain* rates, $\underline{\underline{\chi}}$ the virtual *curvature strain* rates and $\underline{\gamma}$ the virtual *shear strain* rates.

In the case when the virtual velocity fields $\underline{U}$ and $\underline{\Theta}$ are discontinuous across a line Γ, the above expression (2) should be completed by the following additional line integral: