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An operational calculus-based approach to a general bending theory of nonlocal elastic beams

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ABSTRACT

This paper is concerned with obtaining exact solutions for the bending problem of an elastic nanobeam by using the Lurie's operational method. Within the framework of nonlocal elasticity theory, a general governing equation, capable of capturing the size effect, is first constructed in a systematic and straightforward manner. Then a solution methodology is described. Some explicit solutions involving trigonometric expansions are also presented and compared with other well known beam theories. The results indicate that this general beam theory can provide more accurate results, which can be served as benchmarks for other theoretical or numerical methods.

1. Introduction

The nanomaterials in the form of rods, beams, plates and shells are widely used in cutting-edge fields, such as sensing, communications and energy harvesting. They offer excellent performance from the mechanical, electrical, optical and chemical point of view. However, when the structure scales down to nano-domains, an issue of considerable importance, namely, the size effect arises and becomes prominent (Murmu and Adhikari, 2012). The nonlocal elasticity theory, among various higher-order continuum theories that contain additional material length scale parameters, has thus been proposed to develop size-dependent continuum models. With this theory, a nonlocal version of the Euler-Bernoulli beam (EBT) model was initially established by Peddieson et al. (Peddieson et al., 2003) to account for the size effect. In this model, the shear deformation effect was supposed to be neglectable. This assumption may be appropriate for slender beams, but for moderately deep beams, it underestimates the deflection. To remedy this weakness, many suggestions have been made to improve the deflection estimations. Timoshenko beam theory (TBT) (Benzair et al., 2008; Reddy, 2007; Reddy and Pang, 2008; Wang et al., 2008a), Reddy beam theory (RBT) (Reddy, 2007), as well as various higher-order shear deformation beam theories (HOSD) (Aydogdu, 2008, 2009; Thai, 2012; Thai and Vo, 2012) were developed successively. Nevertheless, all these nonlocal beam theories are based on various kinds of hypotheses, which assume the forms of stress or displacement distribution along the thickness. Thus, the applications of these beam theories are also limited due to those approximations inherent in the derivations, which make the solutions inconsistent with all the fundamental equations of nonlocal elasticity. Consequently, more exact theory still deserves great attention.

Operational calculus, stemmed from Heavside's trailblazing contributions (Lindell, 2000; Petrova, 1987), sheds light on this effort, which initiates two routines in developing two and three dimensional exact elastic models. These methods can be collectively referred to as the method of initial functions, by which the unknown stresses and displacements are expressed by initial functions and their derivatives defined in a reference plane. The procedure suggested by Vlasov was in a mixed form (Vlasov, 1957), in which the basic desired stress and displacement are expanded in power series with the expansion coefficients being expressed in terms of initial functions. By means of Sylvester theorem, Das and Setlur extended this method to plane elasto-dynamic problems (Das and Setlur, 1970), while the problem of free vibration of thick circular plates was investigated by Celep (Celep, 1978). Employing this method, some classes of exact solutions for transversely isotropic elastic layers were obtained by Sun and Archer (Sun and Archer, 1992). Along this line, the responses of thin-walled elastic systems, thick plates and shells composed of anisotropic and isotropic materials under static and dynamical actions were

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extensively analyzed. Later, this method was further developed and becoming known as the state space approach, which is often be employed to carry out stress analysis of elastic laminated layers with the concept of transfer matrices.

Alternatively, Lurie developed a symbolic method in a more direct form (Lurie, 2005), by which the *a priori* assumptions regarding the stresses or displacements are excluded. With this method, Cheng (Cheng, 1979) developed a refined theory for elastic plates in which the governing differential equations were deduced directly from the general solution of Boussinesq–Galerkin type. A parallel development of Cheng's theory was made by Barrett and Ellis (Barrett and Ellis, 1988) to discuss the bending of an elastic plate under a transverse load, a new thick plate theory of which was put forward by Wang and Shi (Wang and Shi, 1997) by virtue of the Papkovich–Neuber solution. Additionally, some extensional investigations have been made on the formulation of plate theories, including transversely isotropic piezoelectric plate (Xu and Wang, 2004), thermoelastic plate (Gao and Zhao, 2007), magnetoelastic plate (Gao and Zhao, 2009) and quasicrystalline plate (Gao and Ricoeur, 2012). The analogous development also goes to the formulations of beam theories. The first generalization of Lurie's operational method to anisotropic beam was carried out by Lekhnitskii (Lekhnitskii, 1963). In that study, the elegant solutions were constructed with the aid of the Airy's stress function. Likewise, some refined beam theories established on the basis of general solutions of displacement type (Wang et al., 2008b) were constructed by Gao and his collaborators (Gao, 2009; Gao, 2010; Gao and Wang, 2006a; Gao and Wang, 2006b; Gao et al., 2007).

Nevertheless, to the authors' knowledge, attempts to generalize the formalism to nonlocal elastic beam so as to furnish more accurate solutions have not yet been made. Motivated by these observations, this paper is dedicated to explore such a general theory for a transversely loaded nanobeam within the framework of nonlocal elasticity theory.

2. Basic equations

The main feature of nonlocal elasticity theory is its departure from the traditional postulate of the strictly local internal interactions, but instead it states that the stress state at a reference point $\mathbf{x}$ depends not only on the strain at that point, but also on strains at all other points $\mathbf{x}'$ of the body. In this way, information concerning about the long-range forces between atoms is incorporated into the theory, and consequently, the internal size is represented in constitutive equations simply as a material parameter. The fundamental equations for linear, homogeneous and isotropic nonlocal elastic solids given by Eringen (Eringen, 1983) are as follows

$$\sigma_{ij}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|, \tau)\sigma'_{ij}(\mathbf{x}')\mathrm{d}V(\mathbf{x}'), \tag{1}$$

$$\sigma'_{ij}(\mathbf{x}') = \lambda\varepsilon_{kk}(\mathbf{x}')\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{x}'), \tag{2}$$

$$\varepsilon_{ij}(\mathbf{x}') = \frac{1}{2}\left(\frac{\partial u_j(\mathbf{x}')}{\partial x'_i} + \frac{\partial u_i(\mathbf{x}')}{\partial x'_j}\right), \tag{3}$$

$$\sigma_{ij,j} + \rho(f_i - \ddot{u}_i) = 0. \tag{4}$$

where σ_{ij}, σ'_{ij} and ε_{ij} are, respectively, the nonlocal stress, classical (local) stress and strain tensors. λ and μ are Lamé constants, δ_{ij} the Kronecker delta, V the entire body considered. u_i, ρ, and f_i are, respectively, the displacement vector, mass density and body force density. The nonlocal modulus is represented by the kernel function $\alpha(|\mathbf{x}' - \mathbf{x}|, \tau)$, with τ being defined as small scale factor. Besides, the usual convention of summation over repeated indices i, j and k is utilized in this section, as is the comma convention representing differentiation with respect to the coordinates, and the superposed dot denotes differentiation with respect to time.

The involvement of spatial integrals in constitutive relations (1) makes the theory suffer from theoretical and numerical difficulties. However, Eringen (Eringen, 1983) showed that it is possible to convert this integral relation to an equivalent differential one, namely

$$\left(1 - \beta\nabla^2\right)\sigma_{ij} = \sigma'_{ij} \tag{5}$$

where ∇^2 is the Laplacian operator in $\mathbb{R}^2$, $\beta = (e_0 a)^2$ is the nonlocal parameter that allows for the size effect, a is an internal characteristic length (e.g. length of C–C bond, lattice spacing, granular distance), while e_0 is a constant appropriate to each material, whose magnitude is usually identified either by matching the dispersion curves of plane waves with those of atomic lattice dynamics or by calibrating it against molecular dynamic simulation results. Generally, a conservative estimate for the small scale parameter is suggested as $e_0 a < 2.0$ nm for a single carbon nanotube (Wang and Wang, 2007).

Substituting Eq. (5) into Eq. (4) results in

$$\sigma'_{ij,j} + \rho\left(1 - \beta\nabla^2\right)(f_i - \ddot{u}_i) = 0 \tag{6}$$

Then the equilibrium equation in terms of displacement components can be reached by substituting Eqs. (2) and (3) into Eq. (6)

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \rho\left(1 - \beta\nabla^2\right)(f_i - \ddot{u}_i) = 0. \tag{7}$$

Let us consider the bending of a nonlocal elastic beam as a plane stress problem in a fixed rectangular coordinate system $(x - z)$, where z is the coordinate normal to the neutral surface of the beam. When the body force, as well as the dynamical effect, is absent, this Navier's equation of equilibrium (7) can be simplified as

$$\nabla^2 u_x + \frac{1+\nu}{1-\nu}\frac{\partial\theta}{\partial x} = 0, \quad \nabla^2 u_z + \frac{1+\nu}{1-\nu}\frac{\partial\theta}{\partial z} = 0, \tag{8}$$

where $\theta = \partial u_x/\partial x + \partial u_z/\partial z$ and ν is Poisson's ratio.

The general solution of Eq. (8) has been suggested in various forms (Wang et al., 2008b), the Papkovich–Neuber solution as one of the widely used solutions is adopted herein, namely

$$\begin{aligned} u_x &= P_1 - \frac{1+\nu}{4}\frac{\partial}{\partial x}(P_0 + xP_1 + zP_3), \\ u_z &= P_3 - \frac{1+\nu}{4}\frac{\partial}{\partial z}(P_0 + xP_1 + zP_3), \end{aligned} \tag{9}$$

where the displacement functions $P_i (i = 0,1,3)$ are harmonic and satisfy

$$\nabla^2 P_i = \frac{\partial^2 P_i}{\partial z^2} + \partial_x^2 P_i = 0, \quad \left(i = 0, 1, 3; \quad \partial_x \equiv \frac{\partial}{\partial x}\right). \tag{10}$$

For the bending problem, the nonlocal elastic beam is subjected only to anti-symmetrical loads and boundary conditions, which requires that u_x be an odd function of z and u_z be an even function of z. Then in the light of Lurie's operational method, the symbolic solution of displacements (9) can be rewritten, in terms of the angle of rotation $\psi = [-\partial u_x/\partial z]_{|z=0}$ as well as the deflection w of the neutral plane, as (Gao and Wang, 2006b)