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Secondary bifurcation of a compressible rod with spring-supports

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ABSTRACT

We consider the problem of determining the stability boundary and postbuckling behavior of an elastic rod with spring supports at clamped ends. The rod is loaded by a compressive force and the constitutive equations of the rod take into account the compressibility of the rod axis. Using the Liapunov–Schmidt procedure local bifurcation analysis is performed. The spring stiffness is chosen to be in the neighborhood of the critical one corresponding to a double eigenvalue. Due to the splitting of eigenvalues secondary bifurcations occur. The results show that the type of primary bifurcation depends on the slenderness ratio and that there are three groups of bifurcation diagrams. Also, asymptotic expansions of postbuckling states are constructed.

1. Introduction

The problem of stability analysis of elastic rods supported by elastic springs is an old one. It is well-known that such problems, in order to be fully understood, require evaluation of buckling loads, postbuckling behaviors and the analysis of stability of equilibrium states. Among these problems, the ones having multiple buckling modes associated to the same buckling load (multiple eigenvalues) are of special interest for researchers. Such problems are complex compared to the problems having a simple eigenvalue. However, even more challenging are the problems possessing secondary bifurcations since they could have a very interesting postbuckling behavior that sometimes explains a physical phenomena as mode jumping (see Bauer et al., 1975; Schaeffer and Golubitsky, 1979). In general one can perform local or global analysis of secondary bifurcations (see Domokos, 1994). Here we focus our attention on local analysis near a double eigenvalue. The reason for this is that the authors share a view of Koiter (1976) and Olhoff and Seyranian (2008) and their belief that an understanding of buckling phenomena cannot be achieved without proper knowledge and development of bifurcation theory. There are plenty of papers dealing with secondary bifurcation. One of the most important is the paper Bauer et al. (1975) showing that the splitting of multiple eigenvalues could lead to secondary bifurcation.

When dealing with the secondary bifurcation of plates one must mention the paper Schaeffer and Golubitsky (1979). It is also worth mentioning the papers Buzano (1986) and Wu (1995) concerning the secondary bifurcations of a thin clamped rod under axial compression. In these papers secondary bifurcations lead to a spatially deformed rod. Then there are a lot of papers dealing with the secondary bifurcation of rods supported by elastic springs. Some of them are Potier-Ferry (1983), Wu (1997, 1998a), Wu (1998b) and Hunt and Everall (1999). Following the ideas presented in these papers the present authors find it interesting to generalize the results of Wu (1998a). Namely, lots of papers, analyzing local secondary bifurcation, did not take into account the effect of compressibility. As a consequence we think that the influence of this effect on secondary bifurcation should be investigated. Thus, the whole analysis is limited to the Bernoulli–Euler kinematics meaning that the shear effect will not be taken into account. We note that the postbuckling behavior of compressible rods was the subject of many papers. Among all of them, papers of Antman and co-workers are of great importance. Some of these papers are Antman and Rosenfeld (1978), Antman and Pierce (1990) and Antman and Marlow (1992). It is also worth mentioning the following papers Greenberg (1967), Olmstead and Mescheloff (1974), Atanackovic (1989) and Magnusson et al. (2001).

In this paper the secondary bifurcations of a compressible rod with spring supports at clamped ends is to be analyzed. With the compressibility effect taken into account one may expect complex postbuckling behavior of the rod. The investigation of such complex behavior is supposed to help engineers understand the behavior of elastic structures. Thus the main goal of this paper is a local bifurcation analysis of a straight uniform clamped compressible rod with its ends supported by spring supports of equal stiffness. By applying linear analysis the influence of the slenderness ratio and spring

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stiffness on the lowest buckling load is investigated. The obtained result leads to the conclusion that for some values of the slenderness ratio and spring stiffness the eigenvalue (the lowest buckling load) is double. However, if the spring stiffness is changed the double eigenvalue splits into two simple eigenvalues. Thus the splitting of eigenvalues occurs suggesting that there is a possibility of secondary bifurcation (see Bauer et al., 1975). Using the procedure presented in Shearer (1980) and Wu (1997) the bifurcation analysis is performed showing that primary and secondary bifurcations occur. The type of primary bifurcations (super or subcritical) depends on the slenderness ratio. Further analysis reveals that there are three groups of bifurcation diagrams depending on the slenderness ratio. We note that for an inextensible rod (see Wu, 1998a) only one group of bifurcation diagrams exists. Also, asymptotic expansions of post-buckling states are constructed. At the end of this section we mention that one may find it interesting to investigate the influence of the shear effect. A short analysis of this problem reviles that for the Timoshenko beam the type of primary and secondary bifurcation changes depending on the shear stiffness. However, the postbuckling behavior is different from the one corresponding to a compressible rod.

2. Governing equations

Consider a naturally straight uniform rod BC of length L. The cross-sectional area of the rod is A. The rod is clamped with the ends supported by spring supports of stiffness c (see Fig. 1). We assume that the rod is compressible. In order to derive governing equations we introduce a rectangular Cartesian coordinate system xBy whose axis x coincides with the rod axis in the undeformed state while y axis is perpendicular to x axis (see Fig. 1). The arc length of the rod axis in the undeformed state, measured from the left end B, is denoted by S. The rod is loaded by a compressed force of constant intensity P acting along the x axis (see Fig. 1).

The governing equations describing the behavior of a compressible linearly elastic rod are very well known in the literature (see Atanackovic, 1997; Simitses and Hodges, 2006; Glavardanov et al., 2009). For the rod presented in Fig. 1 these equations are of the form (see Atanackovic, 1997, page 31, Equation 2.1.28)

$$\frac{dV}{dS}=0,\ \frac{dM}{dS}=-(P\sin\theta+V\cos\theta)\left[1+\frac{-P\cos\theta+V\sin\theta}{EA}\right],$$
$$\frac{dy}{dS}=\left(1+\frac{-P\cos\theta+V\sin\theta}{EA}\right)\sin\theta,\ \frac{d\theta}{dS}=\frac{M}{EI},\tag{1}$$

where V is the component of the contact force along the y axis, M is the contact couple, θ is the angle between the tangent to the rod axis and the x axis, E is the modulus of elasticity and I is the second moment of inertia of the cross-section. The corresponding boundary conditions read

$$cy(0)=V(0),\quad -cy(L)=V(L),\quad \theta(0)=0,\quad \theta(L)=0.\tag{2}$$

Assuming that $0<c<\infty$ we introduce the dimensionless quantities

$$t=\frac{S}{L},\quad k=\frac{cL^3}{EI},\quad u=\frac{ycL^2}{EI},\quad \mu=L\sqrt{\frac{A}{I}}$$
$$v=\frac{VL^2}{EI},\quad m=\frac{ML}{EI},\quad \lambda=\frac{PL^2}{EI},\tag{3}$$

so that the system consisting of (1) and (2) becomes

$$v'=0,\ m'=-(\lambda\sin\theta+v\cos\theta)\left[1+\frac{-\lambda\cos\theta+v\sin\theta}{\mu^2}\right],$$
$$u'=k\left(1+\frac{-\lambda\cos\theta+v\sin\theta}{\mu^2}\right)\sin\theta,\ \theta'=m,\tag{4}$$

subject to

$$u(0)=v(0),\quad -u(1)=v(1),\quad \theta(0)=0,\quad \theta(1)=0,\tag{5}$$

where $(\bullet)'=d/dt(\bullet)$. In engineering, the parameter μ is called the slenderness ratio.

The system (4), (5) possesses the trivial solution $v_0=m_0=u_0=\theta_0=0$. If we introduce small perturbations $\Delta v,\Delta m,\Delta u,\Delta\theta$ and express solutions to (4), (5) as $v=v_0+\Delta v=\Delta v$, $m=m_0+\Delta m=\Delta m$, $u=u_0+\Delta u=\Delta u$, $\theta=\theta_0+\Delta\theta=\Delta\theta$ we get the perturbed system being the same as (4), (5) after omitting Δ in front of $\Delta v,\Delta m,\Delta u,\Delta\theta$. Thus the nonlinear system, suitable for bifurcation analysis, is given by (4), (5).

3. Local bifurcation analysis

The aim of this section is to present a local bifurcation analysis based on the nonlinear boundary value problem (4) and (5). In particular, we will be focused on the lowest buckling load, bifurcation equations and primary and secondary branches.

3.1. Buckling loads and bifurcation equations

The first step in local bifurcation analysis is to introduce a vector $\mathbf{w}$ defined by $\mathbf{w}=(w_1,w_2,w_3,w_4)^T=(v,m,u,\theta)^T$ and two function spaces

$$Y=\left\{\mathbf{w}:\mathbf{w}\in C^1([0,1],\mathbb{R}^4),\ \begin{matrix}w_1(0)=w_3(0),\ w_1(1)=-w_3(1),\\ w_4(0)=0,\ w_4(1)=0\end{matrix}\right\},$$
$$Z=\left\{\mathbf{z}:\mathbf{z}\in C([0,1],\mathbb{R}^4)\right\},\tag{6}$$

where $C^1([0,1],\mathbb{R}^4)$ represents the space of continuous functions mapping [0,1] into $\mathbb{R}^4$ and have continuous first derivative, while $C([0,1],\mathbb{R}^4)$ is the space of continuous functions mapping [0,1] into $\mathbb{R}^4$. If the norms $\|\mathbf{w}\|_Y^2=\sum_{i=1}^4(\sup|w_i|+\sup|w_i'|)^2$ and $\|\mathbf{v}\|_Z^2=\sum_{i=1}^4(\sup|v_i|)^2$ are introduced on Y and Z then they become Banach spaces. We can now define the nonlinear operator $\mathbf{F}$: $\mathbb{R}_+\times\mathbb{R}_+\times Y\to Z$

$$\mathbf{F}(\lambda,k,\mathbf{w})=\begin{cases}w_1'\\ w_2'+(\lambda\sin w_4+w_1\cos w_4)\left[1+\dfrac{-\lambda\cos w_4+w_1\sin w_4}{\mu^2}\right]\\ w_3'-k\left(1+\dfrac{-\lambda\cos w_4+w_1\sin w_4}{\mu^2}\right)\sin w_4\\ w_4'-w_2\end{cases}.\tag{7}$$