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Effect of curvature and anisotropy on the finite inflation of a hyperelastic toroidal membrane

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The problem of finite inflation of a hyperelastic toroidal membrane under uniform internal pressure is considered in this paper. The work consists of the following two aspects of the inflation problem. Firstly, a formulation for solving the inflation problem efficiently by directly integrating the differential equations of equilibrium without discretization is proposed. The results obtained are compared with those obtained using a discretization method proposed earlier. Secondly, the effects of the geometric and material parameters of the membrane and the internal pressure on the inflation and its stability are studied. The roles of the curvature (specifically, the eigenvalues of the shape operator) of the toroidal geometry and the membrane material parameter on the distortion of the cross-section and occurrence of wrinkling instability are clearly brought out. Based on the Cauchy stress resultants, the limits on the inflation to avoid wrinkling are determined. It is observed that the limit point pressure of the membrane is inversely proportional to the geometric parameter of the torus. The proportionality constant involved is found to vary linearly with the material parameter of the membrane, and involves two universal constants for the toroidal geometry.

1. Introduction

Inflatable structures have gained considerable importance in the recent past due to their advantages of light weight, quick and self-deployment, and compact storage properties. These structures have potential applications in terrestrial and space structures (Jenkins, 2001). The study of the geometric and material nonlinearities associated with these structures is a challenging issue. The complex interaction of the geometry of the membrane and the associated nonlinearities can result in some non-intuitive behaviour, as evinced in this paper.

Extensive studies have been carried out in the past by many authors on the theory of large elastic deformations (see, for e.g., Adkins and Rivlin, 1952; Green and Adkins, 1970; Green and Zerna, 1992; Rivlin, 1948a,b; Mooney, 1940; Naghdi, 1972; Naghdi and Tang, 1977). Some theoretical analysis on the finite elastic deformations of axisymmetric membranes (circular, cylindrical and spherical) has been done by many authors like Corneliussen and Shield (1961), Needleman (1977), Foster (1967), Hart-Smith and Crisp (1967), Jordan (1962), Sanders and Liepins (1963). Yang and Feng (1970) and Patil and DasGupta (2013) have studied the inflation problem for axisymmetric membranes using an algorithm based on the scale invariance of the equations of equilibrium to efficiently compute the inflated configuration. Eriksson and Nordmark (2012) have studied the parameter dependence in response for the membrane structures using path-following technique in the parameter space. This approach has lead Eriksson (submitted for publication) to formulate and robustly solve the problem of structural optimization with stiffness, strain and instability constraints.

Finite inflation of toroidal membranes has been comparatively less studied. The geometric complexity and variation of curvature on top of the material non-linearity stands in the way of a general analytical solution of the inflation problem. Approximate solutions of the finite inflation of thick and thin-walled toroidal membranes has been presented by Kydoniefs and Spencer (1965), Hill (1980), Kydoniefs and Spencer (1967), Kydoniefs (1967), Feng (1976). The problem has been studied numerically using an iterative Lagrangian perturbation method proposed by Tamadapu and DasGupta (2013). Algorithms proposed for other simply-connected axisymmetric membrane geometries by Yang and Feng (1970), Patil and DasGupta (2013) cannot suitably handle non-simply connected membranes.

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During inflation, membranes with certain geometries/boundary conditions are observed to wrinkle. Wrinkling is a structural instability phenomenon that occurs due to the anisotropic stretching of a membrane. Due to this anisotropic stretching, compressive stresses may develop in the membrane leading to this local buckling (wrinkling) phenomenon. This phenomenon can be related to the non-convexity of the strain energy function of the membrane (which, interestingly, can have a geometric connection). The wrinkling instability has been addressed in a number of studies in the past by many authors like Eftaxiopoulos and Atkinson (2005), Haseganu and Steigmann (1994), Li and Steigmann (1995a, b), Pipkin (1986), Roxburgh (1995), Steigmann (1990). In all these studies, wrinkling has been idealized to be continuously distributed over the membrane. This is a first reasonable approximation in the absence of any bending stiffness of the membrane.

In most of the existing literature, the analysis of the inflation problem of a hyperelastic toroidal membrane has been carried out perturbatively under certain approximations. Finding analytical solutions for these membrane structures is impossible due to the nonlinearity in the material and the kinematics of deformation. The geometry of the membrane, consisting of both positively and negatively curved parts, further aggravates the complexity of the problem. Recently, Tamadapu and DasGupta (2013) have proposed a Ritz based discretization scheme using an iterative Lagrangian perturbation for solving the inflation problem of a toroidal membrane. Their method is very general, and may be applied to an arbitrary membrane geometry. However, no direct approach through integration of the differential equations of equilibrium, similar to that of Yang and Feng (1970), Patil and DasGupta (2013), is available for non-simply connected geometries like the toroidal membrane. Further, the effects of the curvature of the toroidal membrane and material properties on the inflation problem has also not been explored in detail. The issue of wrinkling instability has been addressed in the past, though scarcely. The limit point instability is an intriguing feature of nonlinear hyperelastic membranes (See, e.g., Goriely et al., 2006; Dreyer et al., 1982; Tamadapu et al., 2013). When a membrane passes through the limit point during inflation, it undergoes sudden and rapid stretching (with the possibility of bursting) accompanied with a drop in the gas pressure. The membrane response to pressurization is qualitatively different on the either sides of the limit point. Therefore, it is important to estimate *a priori* the pressure at which such a transition takes place.

Based on the above observations, in this paper, we reconsider the axisymmetric inflation problem of a toroidal membrane of circular cross-section addressed by Tamadapu and DasGupta (2013). However, in contrast to their method (which involves discretization), we reformulate the problem with a different set of suitable field variables and solve it by direct integration of the differential equations of equilibrium of the membrane. Such an approach, though specific to the toroidal geometry, is expected to be more efficient and also applicable to contact and indentation problems involving inflated toroidal membranes. The possibility of above solution is an indication of the solvability of other complicated equations arising from the analysis of axisymmetric membranes. We consider a general axisymmetric deformation field of the material points in the laboratory frame. The membrane material is assumed to be a transversely isotropic Mooney–Rivlin solid. We obtain a two-point boundary value problem for the membrane which is converted to an initial value problem by constructing an optimization function. The optimization problem is then solved using the Nelder–Meads search technique to find the equilibrium configurations of the membrane. The obtained results are compared with the solutions obtained by the modified Ritz method of Tamadapu and DasGupta (2013). Based on the approach proposed in this paper, we then study and analyse the inflation mechanics based on the curvature of the membrane and the material parameter. We identify a qualitative similarity in the variation of the stretch field with the eigenvalues of the shape operator of the toroidal geometry. Based on this correspondence, we discuss the distortion of the cross-section of the membrane and occurrence of wrinkling. We also study the occurrence of the limit point instability of the membrane. It is found that limit point pressure of the membrane is inversely proportional to the geometric parameter (which decides the curvature). The proportionality constant involved, surprisingly, varies linearly with the material parameter involving two universal constants. This functional relation is an invariant property of the toroidal geometry for the Mooney–Rivlin class of hyperelastic materials. The universal constants are specific to the toroidal geometry, and do not depend on any other parameter of the problem. Some interesting insights have been developed through these results.

The paper is organized as follows. In Section 2, we discuss the geometry of deformation of the toroidal membrane. The governing equations of equilibrium are derived using the variational formulation in Section 3. The solution procedure is presented in Section 4, and the numerical results are analysed in Section 5. The paper is concluded with Section 6.

2. Geometry of membrane deformation

Consider a hyperelastic toroidal membrane of circular cross-section as shown Fig. 1 with undeformed ring radius R, sectional radius $\tilde{r}$ and thickness h. The thickness of the membrane is assumed to be small when compared to the characteristic dimensions of the torus. Let θ (meridional) and ϕ (circumferential) be the coordinates on the surface of the torus, as shown in the figure. Let ξ is the coordinate along the local normal with $\xi = 0$ representing the mid-surface of the membrane. The line element on the surface of the undeformed torus can be written as

$$ds^2 = \tilde{r}^2 d\theta^2 + (R + \tilde{r}\cos\theta)^2 d\phi^2 + d\xi^2.$$

Therefore, the components of the undeformed metric tensor are given by

$$g_{ij} = \begin{pmatrix} \tilde{r}^2 & 0 & 0 \\ 0 & (R+\tilde{r}\cos\theta)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{ij} = \left(g_{ij}\right)^{-1}.$$

We assume that Y^3 and $Y^1 - Y^2$ remain, respectively, the axis and the plane of symmetry during the inflation of the membrane. Let the material point $\mathcal{B}$ (deflected from $\mathcal{B}_0$) on the mid-surface of the membrane be represented by $(\tilde{\rho}, \tilde{\eta})$ in the laboratory frame, as shown in Fig. 1. The position vector of a point on the deformed torus is given by

$$p^i = y^i + \xi\lambda_3 n^i. \quad (1)$$

Here,

$$y^1 = \tilde{\rho}(\theta)\cos\phi, \quad y^2 = \tilde{\rho}(\theta)\sin\phi, \quad y^3 = \tilde{\eta}(\theta), \quad (2)$$

and n^i is the unit outward normal vector to the deformed membrane surface given by

$$n^i = \frac{1}{2}\delta^{il}\varepsilon^{\alpha\beta}\varepsilon_{ljk}y^j_{,\alpha}y^k_{,\beta}, \quad \text{(summation convention)} \quad (3)$$

where $\varepsilon^{\alpha\beta} = e^{\alpha\beta}/\sqrt{G}$ and $\varepsilon_{ijk} = e_{ijk}$ are, respectively, the completely antisymmetric contravariant (two dimensional) and covariant (three dimensional) Levi–Civita tensors. Here,