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Eshebly tensors for a finite spherical domain with an axisymmetric inclusion

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ABSTRACT

In recent papers the finite Eshelby tensors for a concentrically placed spherical inclusion in a finite spherical domain have been computed and applied to numerous micromechanical problems. The present work is the extension of the computation of finite Eshelby tensors to general inclusions that are axisymmetric with respect to enclosing spherical domain. The problem of finding the finite Eshelby tensors is transformed into the integral equation. It is shown in the paper that the integral equation has a unique solution. Existence of the solution is proved by exploiting the symmetry of the problem which induce invariant subspaces of the integral equation. In the particular case for a excentrically placed spherical inclusion the problem is explicitly solved. Using computer algebra the solution is found in a closed form up to the second order.

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1. Introduction

Many fundamental problems of the micromechanics require solution of the Eshelby's homogeneous inclusion problem. In engineering applications an inclusion is embedded in a finite domain. Often, there are several inclusions at a high concentration ratio what causes interaction between them. For such problems the dilute suspension model, which is based on a single inclusion within an infinite matrix, is not valid. A more refined theory is required which captures the boundary effect and allows several inclusions within the representative volume (RVE). As established by Li et al. (2007-b), this can be achieved using the notion of the finite Eshelby tensors. They can capture the boundary effect of a RVE and thus the size effect of the different phases.

Inclusion problems for a finite domain have been considered before the seminal paper of Li et al. (2007-a), see e.g. (Kinoshita and Mura, 1984). A first partial solution for a spherical inclusion in a finite spherical domain has been found by Luo and Weng (1987). However, the explicit expressions for the Eshelby tensors were not given. They first appeared in Li et al. (2007-a). Here the Eshelby tensors were obtained for the Dirichlet and Neumann boundary conditions imposed on the boundary of the domain. They were named as the Dirichlet-Eshelby and Neumann-Eshelby tensors. Recently, the notion of the finite Eshelby tensors was further generalized to the composite Eshelby tensors, see Sauer et al. (2008). The finite Eshelby tensors were successfully applied in the three-phase model and the improved Mori-Tanaka theory, see e.g. (Nemat-Nasser and Hori, 1999; Li and Wang, 2008).

So far, in all contributions pertaining to the finite Eshelby tensors, it was assumed that the spherical inclusion is placed concentrically within the spherical RVE. This assumption is removed in the present paper and the inclusion is allowed to be placed axisymetrically with respect to the spherical domain. A particular example of such case is the excentrically placed spherical inclusion. In the first part of the paper the problem of finding the finite Eshelby tensors is considered from the mathematical point of view. Using the well known results from the potential theory the problem is transformed into the integral equation. The unique existence of the problem is proved in three steps. First it is proved that if solution exists, it is unique. The next step is that the integral operator has closed range. Up to this stage presentation is rather standard with the exception that the integral equation is the forth order tensorial equation. The third step deviates and shows that the infinite Eshelby tensor for the inclusion is in the range of the integral equation. This is shown using the symmetry properties of the inclusion and the integral operator. In fact it is proved that the integral equation has finite dimensional invariant subspaces that form a dense subset in an appropriate solution space. The deviation from the standard approach (Dahlberg et al., 1988) is necessary as it is not clear whether the singularity of the integral operator can be canceled by subtracting form it an appropriate adjoint operator. In the case of the Lamé system the transpose of the integral operator cancels it out.

In the second part of the paper the finite Eshelby tensors are computed for the excentrically placed spherical inclusion. The

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solution is found by means of the power series expansion of the infinite Eshelby tensor with respect to the excentricity parameter. Finite dimensional invariant subspaces are explicitly constructed. Some illustrative examples which show convergence of the power series expansion and effects of the excentricy are given. Results are given only for the exterior solution. Using formulae in the paper, the interior solution is easily obtained.

The paper is concluded with a list of possible generalizations. No concrete applications of the new finite Eshelby tensors are given. They will appear in a separate paper. Presentation is mostly restricted to the Dirichlet-Eshelby tensor. Similar theoretical results hold for the Neumann-Eshelby tensor.

2. Notation preliminaries

Throughout the paper tensorial notation is used. Vectors, second, third and fourth order tensors are denoted by $\underline{a}$, $\underline{\underline{a}}$, $\underline{A}$ and $\underline{\underline{A}}$. In component notation with respect to the Cartesian basis vectors $\underline{e}_i$ they are $\underline{a} = a_i\underline{e}_i$, $\underline{\underline{a}} = a_{ij}\underline{e}_i\otimes\underline{e}_j$, $\underline{A} = A_{ijk}\underline{e}_i\otimes\underline{e}_j\otimes\underline{e}_k$ and $\underline{\underline{A}} = A_{ijkl}\underline{e}_i\otimes\underline{e}_j\otimes\underline{e}_k\otimes\underline{e}_l$. Here and in the following the summation convention over the repeated indices is used. A generic notation for a tensor filed of any order is $\mathbf{t}$.

The identity second order tensor is denoted by $\underline{\underline{i}}$ and the fourth order identity tensor by $\underline{\underline{I}}$. The fourth order symmetric unit tensor is

$$\underline{\underline{I}}^s := \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\underline{e}_i\otimes\underline{e}_j\otimes\underline{e}_k\otimes\underline{e}_l$$

A set of orthogonal second order tensors is denoted by $\mathcal{O}(3)$. Symmetric tensor product of two vectors $\underline{a}$ and $\underline{b}$ is denoted by sym $(\underline{a}\otimes\underline{b}) = \frac{1}{2}(\underline{a}\otimes\underline{b} + \underline{b}\otimes\underline{a})$. Dot product of two tensors or single contraction is denoted by a single dot, for example, dot product of a vector $\underline{a}$ and a third order tensor $\underline{A}$ is denoted by $\underline{a}\cdot\underline{A}$ and is defined by $\underline{a}\cdot\underline{A} = a_iA_{ijk}\underline{e}_j\otimes\underline{e}_k$. Note the order of tensors in the dot product. Thus $\underline{A}\cdot\underline{a} = A_{ijk}a_k\underline{e}_i\otimes\underline{e}_j$. Double contraction is denoted by a colon, triple by a dot and column and quadruple contraction by a double colon. For example, $\underline{\underline{A}} : \underline{\underline{B}} = A_{ijkl}B_{klmn}\underline{e}_i\otimes\underline{e}_j\otimes\underline{e}_m\otimes\underline{e}_n$, $\underline{A}\cdot:\underline{B} = A_{ijk}B_{ijk}$ and $\underline{\underline{A}} :: \underline{\underline{B}} = A_{ijkl}B_{ijkl}$. Symmetrization of a tensor $\mathbf{t}$ with respect to indices i and j is denoted by $\mathrm{sym}_{ij}\mathbf{t}$. In particular $\mathrm{sym}_{23}\underline{A} = A_{ijk}\underline{e}_i\otimes\mathrm{sym}(\underline{e}_j\otimes\underline{e}_k)$. Transposition of a tensor $\mathbf{t}$ with respect to indices i and j is denoted by $\mathrm{tran}_{ij}\mathbf{t}$. Thus $\mathrm{tran}_{24}\underline{\underline{A}} = A_{ijkl}\underline{e}_i\otimes\underline{e}_l\otimes\underline{e}_k\otimes\underline{e}_j$. If $\mathrm{tran}_{ij}\mathbf{t} = \mathbf{t}$, we say that $\mathbf{t}$ is $i\leftrightarrow j$ symmetric. Symmetric fourth order tensors have $1\leftrightarrow 2$ and $3\leftrightarrow 4$ symmetry. A symmetric part of a second order tensor $\underline{\underline{a}}$ is denoted by sym $\underline{\underline{a}}$.

Gradient of a tensor filed $\mathbf{t} = \mathbf{t}(X)$ is given by grad $\mathbf{t} = \partial\mathbf{t}/\partial X$. In Cartesian coordinates x_i we have grad $\mathbf{t} = \partial\mathbf{t}/\partial x_i\otimes\underline{e}_i$. Divergence of a tensor field is given as $\mathbf{t} = \mathbf{t} : \underline{\underline{i}}$. For example, $\underline{a} = a_{ij,j}\underline{e}_i$ where the index j after the comma denotes partial differentiation with respect to x_j. Finally, let $\Omega\subset\mathbb{R}^3$ be a domain with the boundary Γ with the exterior normal $\underline{n}$. For a tensor filed $\mathbf{t}$ defined in a neighborhood of $P\in\Gamma$ we denote

$$\mathbf{t}_\pm = \lim_{t\to 0+}\mathbf{t}(P\mp t\,\underline{n}), \tag{1}$$

if the limits exists. Thus $\mathbf{t}_+$ is a limit from the interior and $\mathbf{t}_-$ from the exterior. Moreover, we denote $\Omega_- = \mathbb{R}^3\setminus\overline{\Omega}$.

3. Formulation of the problem

Let $\Omega\subset\mathbb{R}^3$ be a domain and Ω_i an inclusion within Ω. A constant eigenstrain

$$\underline{\underline{\epsilon}}^*(X) = \begin{cases}\underline{\underline{\epsilon}}^* : X\in\Omega_\mathrm{i}\\ \underline{\underline{0}} : X\in\Omega_\mathrm{m} = \Omega\setminus\Omega_\mathrm{i}\end{cases}$$

is prescribed inside the inclusion. Thus $\underline{\underline{\epsilon}}^*(X) = \underline{\underline{\epsilon}}^*\chi(\Omega_i)(X)$ where $\chi(\Omega_\mathrm{i})(X)$ is the characteristic function of Ω_i. An equilibrium displacement vector field $\underline{u}$ with appropriate boundary conditions is sought such that

$$\mathrm{div}\,\underline{\underline{C}} : \mathrm{grad}\;\underline{u} = \mathrm{div}\,\underline{\underline{C}} : \underline{\underline{\epsilon}}^*(X), \tag{2}$$

where $\underline{\underline{C}}$ is a constant elasticity tensor. Since the eigenstrain is discontinuous across the boundary of the inclusion, the above partial differential equation (PDE) should be understood in the distributional sense. It is required that the displacement field $\underline{u}$ and the traction $\underline{t} = (\mathrm{grad}\,\underline{u} - \underline{\underline{\epsilon}}^*(X)) : \underline{\underline{C}}\cdot\underline{n}$ of the total strain field $\mathrm{grad}\,\underline{u} - \underline{\underline{\epsilon}}^*(X)$, are continuous across the inclusion boundary Γ_i. Here $\underline{n}$ is the outward normal to $\partial\Omega_\mathrm{i}$. Thus

$$\begin{aligned}\underline{u}|_+ &= \underline{u}|_-\text{on }\partial\Omega_\mathrm{i},\\ (\mathrm{grad}\,\underline{u} - \underline{\underline{\epsilon}}^*) : \underline{\underline{C}}\cdot\underline{n}|_+ &= \mathrm{grad}\,\underline{u} : \underline{\underline{C}}\cdot\underline{n}|_-\text{on }\partial\Omega_\mathrm{i}.\end{aligned} \tag{3}$$

PDE (2) with (3) and appropriate boundary conditions on the boundary $\partial\Omega$ of Ω constitutes a transmission boundary value problem.

It follows form (3) that the Somigliana identity applies. Thus

$$\begin{aligned}\underline{u}(X) = &\int_{\Omega_i}\frac{\partial\underline{\underline{g}}}{\partial Y} : \underline{\underline{C}} : \underline{\underline{\epsilon}}^*\mathrm{d}\Omega(Y) + \int_{\partial\Omega}\underline{\underline{g}}\otimes\underline{n} : \underline{\underline{C}} : \frac{\partial u}{\partial Y}\mathrm{d}S(Y)\\ &-\int_{\partial\Omega}\frac{\partial\underline{\underline{g}}}{\partial Y} : \underline{\underline{C}} : \underline{u}\otimes\underline{n}\,\mathrm{d}S(Y),\end{aligned} \tag{4}$$

where $\underline{\underline{g}} = \underline{\underline{g}}(X,Y)$ is the Green function and $\underline{n}$ is the outward normal to $\partial\Omega$.

Although more general boundary conditions could be imposed, see Sauer et al. (2008), only two types of the boundary conditions (BC) are considered, Dirichlet BC

$$\underline{u}^0 = \underline{\underline{\epsilon}}^0\cdot(X - O)\text{ on }\partial\Omega \tag{5}$$

and Neumann BC

$$\underline{t}^0 = \underline{\underline{\sigma}}^0\cdot\underline{n}\text{ on }\partial\Omega \tag{6}$$

Here $\underline{\underline{\epsilon}}^0$ and $\underline{\underline{\sigma}}^0$ are the prescribed constant background strain and stress fields. In (5) O is an arbitrary point in $\mathbb{R}^3$. In the case of the Neumann BC we define $\underline{u}^0 = (\underline{\underline{C}}^{-1} : \underline{\underline{\sigma}}^0)\cdot(X - 0)$. Due to the prescribed eigenstrain the solution is sought in the form $\underline{u} = \underline{u}^0 + \underline{u}^\mathrm{d}$ where $\underline{u}^\mathrm{d}$ is the unknown disturbance displacement field. Obviously it solves (2) with the homogeneous boundary conditions on $\partial\Omega$. Equation (2) with Dirichlet or Neumann BC is termed Dirichlet-Eshelby or Neumann-Eshelby boundary value problem (BVP). Using (4) it follows then that in the case of Dirichlet BC the disturbance displacement is given by

$$\underline{u}^\mathrm{d}(X) = \int_{\Omega_\mathrm{i}}\frac{\partial\underline{\underline{g}}}{\partial Y} : \underline{\underline{C}}\,\mathrm{d}\Omega(Y) : \underline{\underline{\epsilon}}^* + \int_{\partial\Omega}\underline{\underline{g}}\otimes\underline{n} : \underline{\underline{C}} : \underline{\underline{\epsilon}}^\mathrm{d}\mathrm{d}S(Y) \tag{7}$$

and in the case of Neumann BC by

$$\underline{u}^\mathrm{d}(X) = \int_{\Omega_\mathrm{i}}\frac{\partial\underline{\underline{g}}}{\partial Y} : \underline{\underline{C}}\,\mathrm{d}\Omega(Y) : \underline{\underline{\epsilon}}^* - \int_{\partial\Omega}\frac{\partial\underline{\underline{g}}}{\partial Y} : \underline{\underline{C}} : \underline{u}^\mathrm{d}\otimes\underline{n}\,\mathrm{d}S(Y) \tag{8}$$

Since (2) is a linear equation and is linear in $\underline{\underline{\epsilon}}^*$, their solutions depend linearly upon $\underline{\underline{\epsilon}}^*$. Moreover a solution space of a BVP with zero boundary data is also linear. Therefore $\underline{u}^\mathrm{d}$ is linear in $\underline{\underline{\epsilon}}^*$ and thus

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