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Nonlinear strain gradient elastic thin shallow shells

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The governing equilibrium equations for strain gradient elastic thin shallow shells are derived, considering nonlinear strains and linear constitutive strain gradient elastic relations. Adopting Kirchhoff's theory of thin shallow structures, the equilibrium equations, along with the boundary conditions, are formulated through a variational procedure. It turns out that new terms are introduced, indicating the importance of the cross-section area in bending of thin plates. Those terms are missing from the existing strain gradient shallow thin shell theories. Those terms highly increase the stiffness of the structures. When the curvature of the shallow shell becomes zero, the governing equilibrium for the plates is derived.

1. Introduction

Thin plate theory has found a lot of applications in the areas of micromechanics and nanomechanics. Thin films, micro-electro-mechanical systems and nano-electromechanical systems are typical applications of the thin beam theory, where size effects have been observed. Many researchers, Papargyri-Beskou et al. (2003), Lazopoulos (2004), have correlated thin beam theory with the strain gradient elasticity theories (Mindlin, 1965; Altan and Aifantis, 1997; Ru and Aifantis, 1993; Yang et al., 2002). The theory of gradient strain elasticity has been applied to many mechanics problems in plasticity and dislocation, (Aifantis, 2003; Fleck and Hutchinson, 1997, 1993, Fleck et al., 1994). Further applications of the strain gradient elasticity theories have appeared in lifting various singularities in fracture problems, Altan and Aifantis (1997) and around concentrated forces like the Flamant problem, Lazar and Maugin (2006).

In the present work the bending Kirchhoff's plate theory will be discussed into the context of a simplified strain gradient elasticity theory, where new terms, depending not only on the moment of inertia of the cross-section but also on the area of the cross-section are introduced. Those terms highly increase the stiffness of the plate. The author, Lazopoulos and Lazopoulos (2010), has already studied the behavior of thin strain gradient elastic beams using the proposed procedure. Terms of the same type have been introduced in bending of beams by Yang et al. (2002) and their theory has been applied to various bending problems, (Lam et al., 2003; Park and Gao, 2006; Ma et al., 2008). Nevertheless, that couple stress theory does not include a substantial part of the strain gradient theory that is the increase of the higher order derivatives in the governing equilibrium equations. Those terms are necessary for the development of boundary layers which are characteristic of the strain gradient elasticity applications. Furthermore Yang et al. (2002) ends up with a symmetric stress tensor assuming zero couple moment, Eq. (33). This requirement is an additional condition which is not derived by any principle of mechanics. Further, couple stresses and symmetric stress tensor is not compatible. In fact the present theory bridges the theories bending theories presented by Papargyri-Beskou et al. (2003) and Yang et al. (2002) in a consistent way including not only the higher order derivatives in the governing equilibrium equations, necessary for the development of boundary layers missing from the theory of Yang et al. (2002), but also the terms depending upon the cross-section area missing from the theory of Papargyri-Beskou et al. (2003), that highly increase the stiffness of the thin beam when the beam thickness reduces. The governing equilibrium equation for the thin plate with the corresponding boundary conditions will be derived through a variational approach for plate bending problems.

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2. Geometrically nonlinear deformations of a shallow thin shell

Adopting Kirchhoff's theory for thin shallow shells along with the nonlinear strain tensor, a simple version of Mindlin's strain gradient elastic constitutive relations is recalled, introducing a geometrically nonlinear theory of elasticity with microstructure, a micro-elasticity theory equipped with two additional constitutive coefficients, apart from the Lame' constants is used. The intrinsic bulk length g and the directional surface energy length l_k are the additional constitutive parameters.

Hence, the strain energy density function, for the present geometrically nonlinear case, is expressed by,

$$W = \tfrac{1}{2}\lambda e_{mm}e_{nn} + Ge_{mn}e_{nm} + g^2\left(\tfrac{1}{2}\lambda e_{kmm}e_{knn} + Ge_{kmn}e_{knm}\right) + l_k\left(\tfrac{1}{2}\lambda(e_{kmm}e_{nn} + e_{mm}e_{knn}) + G(e_{kmn}e_{nm} + e_{mn}e_{knm})\right) \quad (1)$$

where, e_{ij} denotes Green's (or Lagrangean) strain and e_{ijk} the nonlinear strain gradient respectively, with

$$e_{ij} = e_{ji} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_k \cdot \partial_j u_k)\ ,\ e_{ijk} = e_{ikj} = \partial_i e_{kj} \quad (2)$$

and $u_i = u_i(x_k)$, the finite displacement field. The present form of the strain energy density function is the simplest one for the strain gradient elasticity problems including surface energy density, see Vardoulakis (2004).

If the shallow shell is described by the middle surface in its initial shape by the function $z(x,y)$, recalling Kirchhoff's theory of thin shells, the components of the nonlinear Green's tensor are expressed by,

$$\begin{aligned} e_{xx} &= u_x + z_x w_x - \zeta\, w_{xx} + \tfrac{1}{2}w_x^2 \\ e_{yy} &= v_y + z_y w_y - \zeta w_{yy} + \tfrac{1}{2}w_y^2 \\ e_{xy} &= \frac{1}{2}(u_y + v_x)\ + \frac{1}{2}(z_x w_x + z_y w_y)\ - \zeta w_{xy} + \frac{1}{2}w_x w_y \end{aligned} \quad (3)$$

where, (x,y) is the horizontal plane and $w(x,y)$ is the vertical displacement of the point lying on the middle surface. The second Piola–Kirchhoff's stress S_{ij} used in Lagrangean description is defined by,

$$S_{ij} = \frac{\partial W}{\partial e_{ij}} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij} + l_k\left(\lambda e_{knn}\delta_{ij} + 2Ge_{kij}\right), k = x\,\text{or}\,y \quad (4)$$

and the double second Piola–Kirchhoff stresses by,

$$S_{ijk} = \frac{\partial W}{\partial e_{ijk}} = g^2\left(\lambda e_{inn}\delta_{jk} + 2Ge_{ijk}\right) + l_i\left(\lambda e_{nn}\delta_{jk} + 2Ge_{jk}\right) \quad (5)$$

For the present study we consider a thin plate of thickness h shown in Fig. 1. The xy-plane is the plane of the plate, whereas the z axis is the deflection axis. The region of the plate in the xy plane is S_m and the boundary in the xy plane is C. The plate is bending under the action of the distributed transversal loads $p(x,y)$, the edge moments $\overline{M}_{cd}$ and the double moments $\overline{m}_{cd}$ where $c, d = v$ or s, the edge force $\overline{V}_\nu$, exhibiting the (additional) deflection $w(x,y)$ in the ζ-direction.

Therefore, the variation of the strain energy δU of the plate is defined by,

$$\delta U = \iiint_V \left(S_{ij}\delta e_{ij} + S_{ijk}\delta e_{ijk}\right)d\nu \quad (6)$$

It is pointed out that in the existing theories for thin structures into the context of strain gradient elasticity, the contribution of the e_{zij} terms does not exist (Papargyri-Beskou et al., 2003; Papargyri-Beskou and Beskos, 2008; Park and Gao, 2006; Yang et al., 2002). In the present theory, those terms are quite important for thin structures when the thickness of the thin structures is comparable to the bulk intrinsic length of the material. In this case the variation of the strain energy density is expressed by,

$$\begin{aligned}\delta U = \iiint_V &\big((S_{xx}\delta e_{xx} + S_{yy}\delta e_{yy} + 2S_{xy}\delta e_{xy}) + (S_{xxx}\delta e_{xxx} \\ &+ S_{yxx}\delta e_{yxx} + S_{zxx}\delta e_{zxx}) + (S_{xyy}\delta e_{xyy} + S_{yyy}\delta e_{yyy} \\ &+ S_{zyy}\delta e_{zyy}) + 2(S_{xxy}\delta e_{xxy} + S_{yxy}\delta e_{yxy} + S_{zxy}\delta e_{zxy})\big)\mathrm{d}x\mathrm{d}y\mathrm{d}\zeta\end{aligned} \quad (7)$$

Since the shell is thin and shallow, the transverse normal stress S_{zz} may be neglected and the (x, y, ζ) coordinate system can be considered approximately locally rectangular Cartesian. Consequently, we may have

$$\begin{aligned} S_{xx} &= \frac{E}{(1-\nu^2)}(e_{xx} + \nu e_{yy})\ \ ,\ S_{yy} = \frac{E}{(1-\nu^2)}(\nu e_{xx} + e_{yy})\ , \\ S_{xy} &= 2Ge_{xy} \end{aligned} \quad (8)$$

with E Young's modulus, ν Poisson's ratio and G shear modulus.

For the thin shallow shell, the external forces are the body forces prescribed per unit area of the (x,y) plane and their components in the x, y, z directions are denoted by, $\overline{X}, \overline{Y}, \overline{Z}$ correspondingly. The traction per unit length of the boundary C is composed by the forces R_x, R_y, R_z, acting along the x, y, z directions respectively and the double forces R_{xx}, R_{yy}, R_{xxy}, R_{yxy}. Further, the moments $\overline{M}_\nu$, $\overline{M}_s$ are the applied moments per unit boundary length in the normal (ν) and the tangential (S) directions. Non-classical double moments $\overline{m}_{\nu\nu}, \overline{m}_{ss}, \overline{m}_{\nu s}$, due to the gradient elasticity, are also applied to the boundary. Therefore the principle of virtual work gives,

$$\begin{aligned}\delta V = \iiint_V \big\{ &(S_{xx}\delta e_{xx} + S_{yy}\delta e_{yy} + 2S_{xy}\delta e_{xy}) + (S_{xxx}\delta e_{xxx} + S_{yxx}\delta e_{yxx} + S_{zxx}\delta e_{zxx}) + (S_{xyy}\delta e_{xyy} + S_{yyy}\delta e_{yyy} + S_{zyy}\delta e_{zyy}) \\ &+ 2(S_{xxy}\delta e_{xxy} + S_{yxy}\delta e_{yxy} + S_{zxy}\delta e_{zxy})\big\}\mathrm{d}x\mathrm{d}y\mathrm{d}\zeta - \iint_{S_m}[\overline{X}\delta x + \overline{Y}\delta y + \overline{Z}\delta w]\mathrm{d}x\mathrm{d}y - \oint_C \Big\{R_x\delta u + R_y\,\delta v + R_{xx}\delta u_x + R_{yy}\delta v_y + R_{xy}\frac{(\delta u_y + \delta v_x)}{2} \\ &+ R_z\delta w + \overline{M}_\nu\delta w_\nu + \overline{M}_s\delta w_s + \overline{m}_{\nu\nu}\delta w_{\nu\nu} + \overline{m}_{ss}\delta w_{ss} + \overline{m}_{\nu s}\delta w_{\nu s}\Big\}\mathrm{d}s\end{aligned} \quad (9)$$